Differential geometry of generalized almost quaterniohic structures, I

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Abstract

The fibre bundles adjoint to generalized almost quaternionic structures are studied. The most important classes of generalized almost quaternionic manifolds are considered.

Contents

The important role of almost quaternionic structures in the row of differential-geometric structures studied at present is explained primarily by the fact that geometry of almost quaternionic manifolds generalizes the most essential features of geometry of 4-dimensional oriented (pseudo-)Riemannian manifolds being of great importance in theoretical and mathematical physics. Among the features we shall stress the existence of self-dual and anti-self-dual 2-forms on 4-dimensional oriented (pseudo-)Riemannian manifolds generating canonical almost quaternionic structures on the manifolds. Geometry of almost quaternionic manifolds, in its turn, is closely related to Einsteinian geometry of manifolds, the latter having been investigated by many outstanding geometers [1]. Another important feature of 4-dimensional (pseudo-)Riemannian geometry is connected with the remarkable discovery of twistor method by R.Penrose. The method was proposed in the mid$-60s$ for solving problems in gravitation theory and appeared to be fruitful in the field theory of Yang-Mills. The famous Penrose's Twistor Programme [2] consists in using twistor correspondence for transforming conformal invariant fields given on Minkowski complex space subsets into objects of complex algebraic geometry (such as holomorphic bundles, cohomologies with coefficients in analytic bundles and others) that are defined on twistor space subsets. In the works by Penrose and other authors the programme was used for gaugh fields, i.e. solving mass-free equations, on Minkowski spaces. Interpretation of duality equations in terms of algebraic bundle over $\mathbf{C}P^3$ with the help of twistor transformations that was mentioned by Ward in 1977 allowed to reduce the problem of classification of instanton solutions to the problem of algebraic geometry. In this way, for instance, there was received the well-known Atiyah-Drinfeld-Hitchin-Manin classification (ADHM-construction) [3], monopol classification of Nam-Hitchin [4], and others. Twistor construction is closely connected with the existence of canonical almost quaternionic structures on 4-dimensional oriented (pseudo-)Riemannian manifolds mentioned above. Besides, S.Salamon [5] and L.Bérard-Bergery [6] showed independently that the construction is naturally transferred onto quaternionic manifolds of random dimension. It allows to reduce the study of quaternionic manifolds to the study of complex manifolds – associated twistor bundle spaces. Using twistor construction S.Salamon proved [7] that compact quaternionic-Kaehler manifold of positive Ricci curvature for which the second Stiefel-Whitney class of bundle of purely imaginary quaternions is equal to zero is isometric to canonical quaternionic projective space $\mathbf{H}P^n$.

The notion of almost quaternionic structure was first found in 1951 in P.Libermann's paper [8], the structure being considered in its narrow sense, as a pair of anti-commuting almost complex structures on manifold. We call such structures almost quaternionic with parallelizable structural bundle (πAQ-structures). In the paper mentioned above as well as in [9] there was solved one of the most important problems of differentional-geometric structure theory, viz. the problem of integrability of πAQ-srtucture. Namely, P.Libermann showed that a πAQ-manifold is integrable if and only if it is locally isomorphic to quater-

nionic affine space. A great contribution to the study of πAQ-structures was made by M.Obata [10] who investigated the structure of affine connections preserving πAQ-structure in parallel translations, as well as properties of transformations preserving πAQ-structure and by S.Ishihara who studied special types of πAQ-manifolds transformations [11].

Soon the reseachers realized that the class of πAQ-structures is too narrow for constructing a self-contained quaternionic geometry: even a quaternionic projective space $\mathbf{H}P^n$ does not possess the structure of such type. So, there was proposed (and it is generally accepted at present) a broader understanding of almost quaternionic structure as a subbundle of tensor bundle of type (1,1) on manifold whose type fibre is a quaternion algebra (or, equivalently, as $GL(n, \mathbf{H} \cdot Sp(1)$-structures on manifold [12]). Some authors called the structures almost quaternionic [13]. But the study of such structures by traditional methods was difficult as the structures are not strictly and globally defined by a given system of tensor fields like, for example, Riemannian, or almost Hermitian, or almost contact structures. Thus, the question of integrability of almost quaternionic structures is not answered at present (unlike πAQ-structures). Moreover, it is not clear yet in what terminology the integrability criterion could be formulated. Nevertheless, several interesting results in this direction were received, for example, Kulkarni theorem asserting that a compact simply-connected integrable quaternionic manifold is isometric to the quaternionic projective space [14].

The most important results in the theory of almost quaternionic manifolds were received for quaternionic, quaternionic-Hermitian, quaternionic-Kaehler and hyper-Kaehler manifolds. Quaternionic manifolds were first considered by S.Salamon [15]. They are quaternionic counterpart of complex manifolds. Nevertheless, their geometry differs considerably from complex geometry, i.e. unlike Kaehler structures, a quaternionic-Kaehler structure is not always integrable. The basic property of quaternionic manifolds is integrability of a canonical almost complex structure on the space of their twistor bundle for manifolds of dimension greater than 4 [7] (in case of 4-dimension its integrability is known to be equivalent to manifold self-duality [16]). M.Berger proved [17] that quaternionic-Kaehler manifold of dimension greater then 4 is an Einsteinian manifold. It is Ricci-flat if and only if it is locally hyper-Kaehler. Otherwise, it is not even locally reducible. The author also showed [18] that compact oriented quaternionic-Kaehler manifold with positive sectional curvature is isometric to the canonical quaternionic projective space. Hyper-Kaehler structures were thoroughly investigated by A.Beaquville [19] who stated their close connection with complex-symplectic structures. At present a collection of examples of hyper-Kaehler manifolds of a random dimension are known [19,20]. On the other hand, we know very few examples of complete non-hyper-Kaehler quaternionic-Kaehler manifolds, all the manifolds being homogeneous. Moreover, D.V.Alekseevskii proved [21] that any compact homogeneous quaternionic-Kaehler manifold is a Riemannian symmetric space. Such spaces are completely classified by J.Wolf

[22]. The question of the existence of non-symmetric quaternionic-Kaehler manifolds was an open one. The first examples of the manifolds were given by D.V.Alekseevskii [23].

As mentioned above S.Salamon and L.Berard-Bergery have found independently a new approach to the study of (almost) quaternionic manifolds generalizing Penrose twistor construction and allowing to reduce the study of quaternionic manifolds to some problems of complex manifold theory. Within the approach the authors received, for example, the following graceful and fundamental characteristics of quaternionic-Kaehler manifolds:

Theorem. (Salamon [7,5], Bérard-Bergery [6]) *Let M be a quaternionic-Kaehler manifold of positive Ricci curvature. Then its twistor space Z admits Kaehler-Einsteinian metric of positive Ricci curvature. With respect to this metric the natural projection $\pi : Z \to M$ is a Riemannian submersion with totally geodesic fibres. In particular, the compact quaternionic-Kaehler manifold of positive Ricci curvature is simply-connected, all its odd Betti numbers being equal to zero.*

Unfortunately, with this approach the questions connected with possibility of generalization on the multidimensional case of self-dual and anti-self-dual 2-form notions being basic in 4-dimensional Riemannian geometry have not been discussed up to now. In particular, the possibility of generalization of fundamental self-dual and anti-self-dual manifolds notions on the case of almost quaternionic manifolds of arbitrary dimension has remained unclear. The role of spinors in geometry of almost quaternionic manifolds has not been studied, neither the interrelation of almost quaternionic structures in the sense of Libermann (very convenient in applying traditional methods) and the accepted broader understanding of almost quaternionic structures mentioned above.

In the present paper the problems mentioned are investigated and as well as the role of geometry of generalized almost quaternionic structures as generalization of three-web geometry also widely discussed at present (see, for example, [24,25]). Namely, the papers on three-web theory give many examples of three-webs having a number of remarkable properties and showing close relations of the theory with other areas of mathematics, such as, algebraic geometry [26], quasi-group and loop theory [27], etc. This gives hope that the theory of almost quaternionic structures of hyperbolic type, or almost antiquaternionic structures, generalizing directly the three-web theory and being part of the theory of generalized almost quaternionic structures studied here will become the object of further serious investigation.

Sections 1 and 2 introduce the notion of generalized almost quaternionic ($AQ_\alpha-$) structure generalizing the notion of almost quaternionic structure of classical type, as well as the notion of almost antiquaternionic structure, in particular, the three-web structure, on the basis of generalized quaternion algebra. We distinguish and investigate the notion of AQ_α-structure of spinor type and AQ_α-structure with parallelizable structural bundle and study the interrelation

between the structure types. The spinor aspects of geometry of spinor type AQ_α-structures are also discussed, in particular, spintensorial algebra of spinor type AQ_α-manifold is constructed.

In Section 3 we innerly add a special connection generalizing Chern connection of three-web theory to every AQ_α-structure with parallelizable srtuctural bundle. In terms of curvature and torsion tensors of the connection we find integrability criteria of AQ_α-structure with parallelizable structural bundles, as well as of its structural endomorphisms and basic distributions. The notion of isoclinic distribution and isoclinic AQ_α-structure is introduced, and the criterion of isoclinic semi-holonomic AQ_α-sructure with parallelizable structural bundle, generalizing the well-known criterion on three-web being isoclinic of M.A.Akivis, is found. We also introduce the notion of isoclinic-geodesic AQ_α-structure with parallelizable structural bundle and find criteria of AQ_α-structure with parallelizable structural bundle being isoclinically geodesic.

The developed theory allows us to construct whole classes of new interesting (as we hope) examples of almost Hermitian and Einsteinian manifolds on a base of canonical πAQ_α-structure on a pull-back of certain manifolds. Explicitly, we show that the pull-back of locally homogeneous naturally reductive manifold carries two-parameter family of almost Hermitian structures of class G_1 in Gray-Hervella classification [28]. The family contains the unique quasi-Kaehlerian structure which is turn out to be nearly Kaehlerian structure. Moreover, a corresponding two-parameter family of pseudo-Riemannian metrics on such manifold contains exactly four Einsteinian metrics, if the manifold is a semisimple Lie group equipped with Killing metric.

We shall continue our research in the second part of this article. This section will be devoted to applications of the developed methods to needs of self-dual geometry and its multidimensional generalization. We shall introduce and investigate the so called vertical type AQ_α-structures which play fundamental role in multidimensional generalization of self-dual geometry. We shall prove that quaternionic-Kaehler structures are the vertical type AQ_α-structures. We shall introduce and investigate generalized quaternionic Kaehler manifolds and show that the manifolds of dimension greater than four are Einsteinian, which generalizes the well-known result of M.Berger [17]. We shall prove the multidimensional generalization of the well-known Atiyah-Hitchin-Synger criterion of 4-dimensional oriented Riemannian manifold being Einsteinian. We shall introduce the notions of twistor curvature tensor and t-conformal semiflat AQ_α-manifold and develop multidimensional generalization of classical self-dual geometry on this base. Finally, we shall consider 4-dimensional conformal semiflat generalized Kaehler manifold and obtain some exhausting results in this direction.

1 Spinor Geometry of Generalized Quaternion Algebra

The characteristic feature of generalized almost quaternionic structures is that in general case giving of the structure on a manifold is not reduced to giving of one or several tensor fields. This fact makes it difficult to directly apply classical methods to its studying from the view point of differential geometry. On the other hand, giving such a structure on a manifold determines the fibre bundle over the manifold, whose standard fibre being the algebra of generalized quaternions. Studying geometry of generalized almost quaternionic structures is closely connected with studying the orbits of linear representation of such algebra. In particular, studying the natural representation of the algebra being us to an interpretation of spinor geometry on the basis of generalized quaternion algebra.

1.1 Algebra of α-quaternions

Let A be an algebra over the field F, $char F \neq 2$, with unit 1 and involution $a \to \overline{a}$ $(a \in A)$, and $a + \overline{a} \in F$, $a \cdot \overline{a} \in F$. Recall [9] that the Cayley-Dickson duplication procedure allows to construct by every nonzero element $\alpha \in F$ new algebra (A, α) which is F-modul $F^2 = F \oplus F$, with the operation of multiplication $(a_1, a_2)(b_1, b_2) = (a_1 b_1 + \alpha b_2 \overline{a}_2, \overline{a}_1 b_2 + b_1 a_2)$. Here A is enclosed in (A, α) as a subalgebra of pairs of the type $\{ (a, 0) \mid a \in A \}$. Denote $i = (0, 1) \in (A, \alpha)$, then $i^2 = \alpha$. Now, by the above identification we have $(A, \alpha) = A \oplus iA$. Note that (A, α) is also an algebra over F with involution: if $x = a_1 + ia_2 \in (A, \alpha)$, then $\overline{x} = \overline{a}_1 - ia_2$. The involution has the same properties: $x + \overline{x} \in F$, $x \cdot \overline{x} \in F \quad (x \in (A, \alpha))$. Moreover, if the quadratic form $|a|^2 = a \cdot \overline{a}$ is non-degenerate on A, then the quadratic form $|x|^2 = x \cdot \overline{x}$ is also non-degenerate on (A, α) (see [9], p.42). This involution is called *the operation of conjugation.*

If $A = F = \mathbf{R}$, then $(\mathbf{R}, \alpha) = \mathbf{R} \oplus i\mathbf{R}$ up to multiplication of i by constant multiplier $\sqrt{|\alpha|}$ is either the field $\mathbf{C}$ of complex numbers $(\alpha = -1)$ or the ring $\mathbf{D}$ of double numbers $(\alpha = 1)$. Denote $(\mathbf{R}, \alpha)$ by $\mathbf{K}_\alpha$.

Applying Cayley-Dickson procedure to $\mathbf{K}_\alpha$ we get the algebra $((\mathbf{R}, \alpha), \beta) = \mathbf{R} \oplus \mathbf{iR} \oplus \mathbf{jR} \oplus \mathbf{kR}$, (where $j = (0, 1) \in (\mathbf{K}_\alpha, \beta), k = i \cdot j$), called *the algebra of generalized quaternions* which is known to be associative and non-commutative [9, p.44]. This algebra up to multiplication of i and j by constants $\sqrt{|\alpha|}$ and $\sqrt{|\beta|}$ (called it calibration), respectively, is either the body $\mathbf{H}$ of quaternions $(\alpha = \beta = -1)$ or the ring $\mathbf{H}'$ of antiquaternions $(\alpha = \beta = 1)$ (the case $\alpha \cdot \beta < 0$ corresponds to the algebra of antiquaternions). As the calibration does not make any essential changes in the properties of generalized quaternion algebra we consider the algebra calibrated and denote $\mathbf{H}_\alpha = ((\mathbf{R}, \alpha), \alpha); \alpha = \pm 1$, We call it *the algebra of α-quaternions.* Note that for α -quaternion $q = a + ib + jc + kd =$

$(a+ib)+j(c-id)$ we have $k^2=-1; \overline{q}=a-ib-jc-kd;\ |q|^2=q\cdot\overline{q}=a^2-\alpha b^2-\alpha c^2+d^2$.

It follows that any α-quaternion $q\in \mathbf{H}_\alpha$ admits a representation of the type $q=z_1+jz_2;\quad z_1,z_2\in \mathbf{K}_\alpha,\ j^2=\alpha$ allowing to construct in the natural way a representation of algebra $\mathbf{H}_\alpha$ into algebra $\operatorname{End}\mathcal{S}$ of endomorphisms of algebra $\mathbf{H}_\alpha$ regarded as $\mathbf{K}_\alpha$-modul $\mathcal{S}=\mathbf{K}_\alpha\oplus\mathbf{K}_\alpha$ that we call *an α-spinvector space.* Namely, let $q=z_1+jz_2\in\mathbf{H}_\alpha, X=(X_1,X_2)\in\mathcal{S}$. Put $[q]X=q\cdot(X_1+jX_2)=(z_1X_1+\alpha\overline{z}_2X_2)+j(z_2X_1+\overline{z}_1X_2)$. Evidently, the matrix of endomorphism $[q]$ in basis $\{1,j\}$ has form:

$$(q)=\begin{pmatrix} z_1 & \alpha\overline{z}_2\\ z_2 & \overline{z}_1\end{pmatrix}. \tag{1}$$

Note that the α-spinvector space $\mathcal{S}$ is innerly endowed with involution $\tau: X\to\overline{X}$ generated by the operator of conjugation of α-quaternions and called *the operation of conjugation of α-spinvectors.* Besides, $\mathcal{S}$ is innerly endowed with Hermitian metric $\langle\langle\cdot,\cdot\rangle\rangle$. Namely, let $X=(z_1,z_2)\in\mathcal{S}$. Denote $z_1=\operatorname{Re}X,\ z_2=\operatorname{Im}X$. Since $\overline{X}=(\overline{z_1},-z_2),\ \operatorname{Re}\overline{X}=\overline{\operatorname{Re}X},\ \operatorname{Im}\overline{X}=-\operatorname{Im}X$. Further, let $Y=(u_1,u_2)$ be another α-spinvector. Put $\langle\langle X,Y\rangle\rangle=\operatorname{Re}(\overline{X}Y)=\overline{z}_1u_1-\alpha\overline{z}_2u_2$. Evidently, this is a non-degenerate Hermitian form, $\langle\langle qX,qY\rangle\rangle=\operatorname{Re}(\overline{qX}\cdot qY)=\operatorname{Re}(\overline{X}\cdot\overline{q}\cdot q\cdot Y)=|q|^2\langle\langle X,Y\rangle\rangle$. Thus, we get

Proposition 1. *Natural representation of non-isotropic really normalized α-quaternions in the space $\mathcal{S}$ is realized by conformal endomorphisms of module $\mathcal{S}$.* $\square$

Note that $|q|^2=q\cdot\overline{q}=(z_1+jz_2)(\overline{z}_1-jz_2)=z_1\overline{z}_1-\alpha z_2\overline{z}_2=\det(q)=\det[q]$. In particular, unit α-quaternions, that in view of algebra $\mathbf{H}_\alpha$ being compositional form a multiplicative subgroup of ring $\operatorname{End}\mathcal{S}$, are realized in this representation as unitary unimodular endomorphisms of module $\mathcal{S}$. Thus, we get

Proposition 2. *Multiplicative subgroup of unit α-quaternions in natural representation is identified with Lie group $SU(2,\mathbf{K}_\alpha)=SP(1)$.* $\square$

Remark. If $\alpha=1$ Lie group $SU(2,\mathbf{D})$ is naturally isomorphic to Lie group $SL(2,\mathbf{R})$. This isomorphism allowing to identify the given Lie groups juxtaposes the matrix

$$\begin{pmatrix} x_1+iy_1 & x_2-iy_2\\ x_2+iy_2 & x_1-iy_1\end{pmatrix}\in SU(2,\mathbf{D})$$

to the matrix

$$\begin{pmatrix} x_1+y_1 & x_2-y_2\\ x_2+y_2 & x_1-y_1\end{pmatrix}\in SL(2,\mathbf{R}).$$

In particular, Lie algebras $\mathsf{su}(2,\mathbf{K}_\alpha)$ are semisimple.

Lie group $SU(2,\mathbf{K}_\alpha)$ is a 3-dimensional Lie group acting on its Lie algebra $\mathsf{su}(2,\mathbf{K}_\alpha)$ by a adjoined representation $\operatorname{Ad}(g)X=gXg^{-1}\quad (g\in SU(2,\mathbf{K}_\alpha)$,

$X \in \mathsf{su}(2, \mathbf{K} - \alpha))$, orthogonal in the Killing form of this Lie algebra. In view of the above and in force of the connectivity of Lie group $SU(2, \mathbf{K}_\alpha)$ the image of mapping Ad lies in Lie group $SO(2-\alpha, 1+\alpha; \mathbf{R})$. Thus, the homomorphism $s = \mathrm{Ad} : SU(2, \mathbf{K}_\alpha) \to SO(2-\alpha, 1+\alpha; \mathbf{R})$ is innerly defined. In view of Lie group $SU(2, \mathbf{K}_\alpha)$ being semisimple the mapping s_* being a adjointed representation of its Lie algebra is non-degenerate and since $\dim \mathsf{su}(2, \mathbf{K}_\alpha) = \dim \mathsf{so}(2-\alpha, 1+\alpha; \mathbf{R}) = 3$, it is an isomorphism of Lie algebra, and therefore, the mapping s-(a two-leaf one) is a covering mapping. The (two-valued) mapping $s^{-1} : SO(3, \mathbf{R}) \to SU(2, \mathbf{C})$ is a classicall spinor representation of an orthogonal group when $\alpha = -1$. In the case of arbitrary algebra $\mathbf{H}_\alpha$ we follow the same notation.

Note that subspace $IQ \subset \mathbf{H}_\alpha$ of purely imaginary α-quaternions is canonically identified with the space of Lie algebra $\mathsf{su}(2-\alpha, 1+\alpha; \mathbf{R})$ that, in its turn, admits canonical identification with Lie algebra $\mathsf{so}(2-\alpha, 1+\alpha; \mathbf{R})$ by means of mapping $s_* = \mathrm{ad} : \mathsf{su}(2, \mathbf{K}_\alpha) \to \mathsf{so}(2-\alpha, 1+\alpha; \mathbf{R})$. With the above identifications and in view of (1) the purely imaginary α-quaternion $q = ia + jb + kc$ has the corresponding matrix

$$(q) = \begin{pmatrix} ia & \alpha(b+ic) \\ b-ic & -ia \end{pmatrix}$$

which is an element of Lie algebra $\mathsf{su}(2, \mathbf{K}_\alpha)$. Also, $[(q_1), (q_2)] = (q_1)(q_2) - (q_2)(q_1)$. Now, in force of the above identification the space IQ acquires the structure of Lie algebra isomorphic to $\mathsf{su}(2, \mathbf{K}_\alpha)$ with commutator $[q_1, q_2] = q_1\, q_2 - q_2\, q_1$. Besides, $s_*(q)X = \mathrm{ad}\,(q)X = (q)X - X(q)$, hence we have:

$$s_*(q) = \begin{pmatrix} 0 & 2\alpha c & -2\alpha b \\ -2\alpha c & 0 & 2\alpha a \\ -2b & 2a & 0 \end{pmatrix}$$

in the basis $\{\, (i), (j), (k)\,\}$. In view of the above it is easy to compute that

$$-\,\mathrm{tr}\,(s_*(q)\, s_*(q)) = -4\,\mathrm{tr}\,((q)(q)) = 8q\overline{q},$$

that is we get the valid

Proposition 3. *The norm of purely imaginary α -quaternions as elements of Lie algebras $\mathsf{su}(2, \mathbf{K}_\alpha)$ or $\mathsf{so}(2-\alpha, 1+\alpha; \mathbf{R})$ in Killing metric differs from their norms as elements of algebra $\mathbf{H}_\alpha$ respectively only by the multiplier 2 or 8.* $\square$

1.2 Spinbases

The norm $|q|^2 = q \cdot \overline{q}$ of α-quaternions allows to regard the algebra $\mathbf{H}_\alpha$ as a real Euclidean space in which scalar product is reconstructed by polarization:

$$\langle q_1, q_2 \rangle = \frac{1}{2}(\overline{q}_1 \cdot q_2 + \overline{q}_2 \cdot q_1); \qquad q_1, q_2 \in \mathbf{H}_\alpha. \tag{2}$$

Note that $\mathrm{Im}\,(\overline{q}_1 \cdot q_2) = \mathrm{Im}\,\overline{(\overline{q}_2 \cdot q_1)} = -\mathrm{Im}\,(\overline{q}_2 \cdot q_1)$, hence $\langle q_1, q_2\rangle = \frac{1}{2}(\mathrm{Re}\,(\overline{q}_1 \cdot q_2) + \mathrm{Re}\,(\overline{q}_2 \cdot q_1))$. Recalling definition of the form $\langle\langle\cdot,\cdot\rangle\rangle$ we get

$$\langle X, Y\rangle = \frac{1}{2}\{\, \langle\langle X, Y\rangle\rangle + \langle\langle Y, X\rangle\rangle \,\}; \qquad X, Y \in \mathcal{S}. \tag{3}$$

Further, it is evident that

$$\langle\langle X, Y\rangle\rangle = \langle X, Y\rangle + i\langle X, i^3 Y\rangle; \qquad X, Y \in \mathcal{S}. \tag{4}$$

Consider (2) again. In particular, let $q_1, q_2 \in IQ$. Then $\overline{q}_1 = -q_1, \overline{q}_2 = -q_2$, and (2) assumes the form

$$q_1 \cdot q_2 + q_2 \cdot q_1 = -2\langle q_1, q_2\rangle; \qquad q_1, q_2 \in IQ. \tag{5}$$

Now we get the following

Proposition 4. *The system $\{j_1, j_2, j_3\} \in IQ$ forms an orthonormal basis of the space IQ iff*

$$j_k \cdot j_m + j_m \cdot j_k = \begin{cases} \pm 2, & \text{for } k = m \\ 0, & \text{for } k \neq m. \end{cases} \qquad \square$$

For example, the system $\{i, j, k\}$ forms an orthonormal basis in IQ. Recall that matrices having these elements as endomorphisms of $\mathbf{K}_\alpha$-module S have (by (1)) the following form:

$$\sigma_1 = (i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \; \sigma_2 = (j) = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}; \; \sigma_3 = (k) = \begin{pmatrix} 0 & \alpha i \\ -i & 0 \end{pmatrix}; \tag{6}$$

In is easy to see that when $\alpha = -1$ the matrices differ from the classical Pauli ones only by multiplier i. We call them it generalized Pauli matrices.

Theorem 1. *For any orthonormal basis $\{\, j_1, j_2, j_3 \,\}$ of the space IQ there exists an orthonormal basis $\{\, \varepsilon_1, \varepsilon_2 \,\}$ of $\mathbf{K}_\alpha$-moduleS, where $(j_1) = \sigma_1, (j_2) = \sigma_2, (j_3) = \pm\sigma_3$. The basis is defined up to the action of group $U(1, \mathbf{K}_\alpha)$ enclosed in $U(2, \mathbf{K}_\alpha)$ by diagonal matrices.*

Proof. Consider vectors $\varepsilon' = \frac{X + i[j_1]^3 X}{|X + i[j_1]^3 X|}$ and $\varepsilon' = \frac{X - i[j_1]^3 X}{|X - i[j_1]^3 X|}$, where $X \in \mathcal{S}$ is any vector not being eigenvector of operator $[j_1]$, i is imaginary unit of ring $\mathbf{K}_\alpha$. Evidently, the vectors are eigenvectors of operator $[j_1]$ having eigenvalues i and $-i$, respectively. In view of eigenspaces of the operator being one-dimensional, each of the vectors is defined up to multiplication by element $z \in \mathbf{K}_\alpha, |z| = 1$. It is clear that in basis $\{\varepsilon'_1, \varepsilon'_2\}$ $\;(j_1) = \sigma_1$. Further, $[j_1]\varepsilon'_1 = i\varepsilon'_1$ and hence $[j_2][j_1]\varepsilon'_1 = i[j_2]\varepsilon'_1$. On the other hand $[j_2][j_1]\varepsilon'_1 = -[j_1][j_2]\varepsilon'_1$, hence $[j_1][j_2]\varepsilon'_1 = -i[j_2]\varepsilon'_1$, i.e. $[j_2]\varepsilon'_1 = a\varepsilon'_1, \quad a \in \mathsf{K}_\alpha$. Similarly, $[j_2]\varepsilon'_2 = b\varepsilon'_1, \quad b \in \mathbf{K}_\alpha$. Thus,

$$(j_2) = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix},$$

and since $(j_2) \in \mathsf{su}(2, \mathbf{K}_\alpha)$, $\overline{a} = ab$. Besides, since $|j_2|^2 = -\alpha$, i.e. $\det[j_2] = -\alpha$, we have $ab = \alpha$, $\quad |a|^2 = a\overline{a} = \alpha ab = a^2 = 1$, $|b|^2 = b\overline{b} = \alpha\overline{a}\overline{b} = 1$. Therefore,

$$(j_2) = \begin{pmatrix} 0 & \alpha\overline{a} \\ a & 0 \end{pmatrix}, \qquad |a|^2 = 1.$$

Assume $\varepsilon_1 = x\varepsilon'_1, \varepsilon_2 = y\varepsilon'_2$, so that in basis $\{\varepsilon_1, \varepsilon_2\}$ we have $(j_2) = \sigma_2$. Now we have: $[j_2]\varepsilon_1 = x[j_2]\varepsilon'_1 = xa\varepsilon'_2 = xay^{-1}\varepsilon_2$; $[j_2]\varepsilon_2 = y[j_2]\varepsilon'_2 = \alpha y\overline{a}\varepsilon'_1 = \alpha y\overline{a}x^{-1}\varepsilon_1$. Thus, x and y must be chosen so that $xy^{-1} = \alpha a^{-1}$. Then, automatically, $\alpha y\overline{a}x^{-1} = (xy^{-1})^{-1}\alpha\overline{a} = \alpha a\alpha\overline{a} = \alpha^2|a|^2 = 1$, $xay^-1 = xy^-1a = \alpha$. Thus, we can choose an arbitrary element of ring $\mathbf{K}_\alpha$ with absolute value 1 as x, and assume $y = \alpha ax$. Then in basis $\{\,\varepsilon_1, \varepsilon_2\,\}$ defined up to multiplication of its elements by $z \in \mathbf{K}_\alpha$, $|z| = 1, (j_2) = \sigma_2$. Finally, we similarly get that in this basis

$$(j_3) = \begin{pmatrix} 0 & \alpha\overline{b} \\ b & 0 \end{pmatrix}, \qquad b \in \mathbf{K}_\alpha, |b|^2 = -\alpha.$$

and in view of $\langle j_2, j_3 \rangle = 0$, and hence $[j_2][j_3] + [j_3][j_2] = 0$, we have:

$$\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \alpha\overline{b} \\ b & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha\overline{b} \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} = 0,$$

hence $\alpha(b + \overline{b}) = 0$, i.e. $b = \pm i$, and $(j_3) = \pm\sigma_3$. $\quad\square$

Definition 1. An orthonormal basis of $\mathbf{K}_\alpha$-module S corresponding to the orthonormal basis of real space of purely imaginary α-quaternions is called *a spinbasis.*

In view of linear connection of Lie groups $SU(2, \mathbf{K}_\alpha)$ we get

Proposition 5. *Orthonormal basis* $\{\,j_1, j_2, j_3\,\}$ *of the space* IQ *has the same orientation as the basis* $\{\,i, j, k\,\}$ *of the same space iff in the spinbasis corresponding to the basis* $\{\,j_1, j_2, j_3\,\}$ *we have*

$$(j_k) = \sigma_k; \qquad k = 1, 2, 3.$$

Proof. Let $g \in SU(2, \mathbf{K}_\alpha)$ be the element transferring spinbasis $\{\,\varepsilon_1^0, \varepsilon_2^0\,\}$, where $(i) = \sigma_1, (j) = \sigma_2, (k) = \sigma_3$ into the spinbasis, where $(j_k) = \sigma_k, k = 1, 2, 3; \omega(t)$ is the path into $SU(2, \mathbf{K}_\alpha)$ connecting unit $e \in SU(2, \mathbf{K}_\alpha)$ with g. Then $s \circ \omega(t)$ is the path in the group space of Lie group $SO(2 - \alpha, 1 + \alpha; \mathbf{R})$ deforming basis $\{\,i, j, k\,\}$ into basis $\{\,j_1, j_2, j_3\,\}$. $\quad\square$

1.3 Linear representation orbits of α-quaternion algebra

Let the precise linear representation of algebra $\mathbf{H}_\alpha$ be fixed. Then the algebra is realized as a subalgebra of algebra $\operatorname{End} V$ of endomorphisms of (real) linear space V. Consider the orbit $Orb(X) = \{\, Y \in V \mid Y = q(X);\ q \in \mathbf{H}_\alpha\}$ of the nonzero element $X \in V$. It can be regarded as the image of homomorphism $X : \mathbf{H}_\alpha \to V;\ X(q) = q(X)$. Evidently, $\ker X \subset \{q \in \mathbf{H}_\alpha \mid |q| = 0\,\} \quad (X \in V)$. This means that when $\ker X \neq \{0\}, X \in \ker q$, where q is an isotropic α-quaternion. Thus, when $X \in V$ does not belong to the kernel of isotropic α-quaternion, $\ker X = \{0\}$ and $\dim Orb(X) = 4$.

In order to study the case when X belongs to the kernel of isotropic α-quaternion we build the following construction. Consider $\mathbf{K}_\alpha$-module $V^{\mathbf{K}_\alpha} = V \otimes_{\mathbf{R}} \mathbf{K}_\alpha$ and the naturally induced linear representation of algebra $\mathbf{H}_\alpha$ on $V^{\mathbf{K}_\alpha}$ that we call *an extended representation* of the algebra. Let $\{\,\mathrm{id}, I, J, K\}$ is orthonormal basis of space $\mathbf{H}_\alpha$. Evidently, $V^{\mathbf{K}_\alpha} = D_I^i \oplus D_I^{-i}$, where D_I^λ is the eigensubmodule of operator I with eigenvalue λ, i is the imaginary unit of ring $\mathbf{K}_\alpha; i^2 = \alpha$. We have $\sigma = \frac{1}{2}(\,\mathrm{id}+iI^3)$ is a projector on D_I^i and $J|D_I^i : D_I^i \to D_I^{-i}$ is an isomorphism of $\mathbf{K}_\alpha$-modules. Let $X \in D_I^i,\ \ q = a\,\mathrm{id} + bI + cJ + dK \in \mathbf{H}_\alpha$. Then $q(X) = (a+bi)X + (c-di)JX$; $q(JX) = \alpha(c+di)X + (a-bi)JX$. Thus $Orb(X) = \mathcal{L}(X, JX)$. In particular, $\dim Orb(X) = 2$, and

$$(q|Orb\,X) = \begin{pmatrix} a+bi & \alpha(c+di) \\ c-di & a-bi \end{pmatrix}.$$

We get

Proposition 6. *Orbits of eigenvectors of α-quaternions with purely imaginary eigenvalues are 2-dimensional and have the natural structure of α-spin-vector space.*

Now, let $X \in V^{\mathbf{K}_\alpha}$ be an arbitrary element. Then $X = a + u, a \in D_I^i, u \in D_I^{-i}$. Thus $Orb(X) = \mathcal{L}(a, Ja, u, Ju) = \mathcal{L}(a, Ju) \oplus \mathcal{L}(Ja, u)$. Indeed, $\mathcal{L}(a, Ju) \subset D_I^i, \mathcal{L}(Ja, u) \subset D_I^{-i}$, and since $D_I^i \cap D_I^{-i} = \{0\}$, we have: $\mathcal{L}(a, Ju) \cap \mathcal{L}(Ja, u) = \{0\}$. Thus, either $\dim Orb(X) = 2$, or $\dim Orb(X) = 4$, and $\dim Orb(X) = 2$ iff vectors $\{a, Ju\}$ are linearly dependent. Therefore, in the former case $Orb(X)$ is the orbit of eigenvector a (or u) having, as shown above, a natural structure of α -spinvector space, and in the latter case $Orb(X)$ is the direct sum of two such subspaces conjugated with respect to J. So, we can have

Definition 2. 2-dimensional orbits of an extended linear representation of α-quaternion algebra are call *spinor subspaces*, its 4-dimensional orbits being *bispinor subspaces* of space $V^{\mathbf{K}_\alpha}$.

Consider, finally, the case when $X \in V$. Then $X = a + \tau a, \quad a = \sigma X$, where $\tau : V^{\mathbf{K}_\alpha} \to V^{\mathbf{K}_\alpha}$ is the operator of natural conjugation. Thus, $Orb(X) = \mathcal{L}(a, Ja, \tau a, J\tau a) = \mathcal{L}(a, J\tau a) \oplus \mathcal{L}(Ja, \tau a)$. Here $\dim Orb(X) = 2$ iff vectors $\{Ja, \tau a\}$ are linearly dependent. Let $Ja = \lambda \tau a, \quad \lambda \in \mathbf{K}_\alpha$. We have $J^2 a =$

$\lambda J\tau a = \lambda\tau(\lambda\tau a) = \lambda\overline{\lambda}a, \quad \lambda \in \mathbf{K}_\alpha$. On the other hand, $J^2 a = \alpha a$. Thus, $|\lambda|^2 = \alpha$. If $\mathbf{K}_\alpha = \mathbf{C}$, then $\alpha = -1$ and this case is impossible. Hence, $\mathbf{K}_\alpha = \mathbf{D}, \alpha = 1, |\lambda| = 1$. Besides, $\tau JA = \overline{\lambda}a$. But $\tau Ja = \frac{1}{2}\tau(JX + iJIX) = \frac{1}{2}(JX - iJIX) = \frac{1}{2}(JX + iIJX)$, so $JX + iIJX = \overline{\lambda}(X + iIX)$, hence $JX = \overline{\lambda}X$ and thus, $\lambda = \pm 1$, that is $X \in D_I^{\pm 1}$. Inversely, if $X \in D_I^{\pm 1}$, then $q(X) = (a \pm b)X + (c \mp d)JX, q(JX) = (c \pm d)X + (a \mp b)JX$. In particularly,

$$\dim Orb(X) = 2, \quad (q) = \begin{pmatrix} a \pm b & c \pm d \\ c \mp d & a \mp b \end{pmatrix}.$$

Thus, the orbit of element $X \in V$ is 2-dimensional iff X is the eigenvector of purely imaginary α-quaternion I. Here $X \in \ker(\mathrm{id} + I)$ or $X \in \ker(\mathrm{id} - I)$, i.e. $X \in \ker q, \; |q| = 0$. Inversely, let $X \in \ker q, \; |q| = 0$. We have $q = a + b\tilde{q}; \quad a, b \in \mathbf{R}, \quad \tilde{q}$ is purely imaginary α-quaternion. Then $q(X) = 0$, i.e. $\tilde{q}(X) = -\frac{a}{b}X; \frac{a}{b} = \pm 1$, thus X is the eigenvector of purely imaginary α -quaternion $\tilde{q}$. It is easy to check that in this case $Orb(X)$ consists of kernel elements of some isotropic α-quaternions. Indeed, if X is eigenvector of purely imaginary α-quaternion I, i.e. $IX = \pm X$; $q = a\,\mathrm{id} + bI + cJ + dK \in \mathbf{H}_\alpha$ is any α-quaternion, then $\{\, \tilde{q} \in \mathbf{H}_\alpha \mid \tilde{q}(qX) = 0 \,\} = \{(u^2 + v^2)x \;\mathrm{id} + (\mp(u^2 - v^2)x - 2uvy)I + (-2uvx \pm u^2 - v^2)y)J + (u^2 + v^2)yK \mid x, y \in \mathbf{R}\,\}$ where $u = a + b, v = c - d$. Therefore, we have proved

Theorem 2. *The dimension of the linear representation orbit of α-quaternion algebra is equal to either 2 or 4. It is 2 iff the orbit belongs to unification of kernels of isotropic α-quaternions.* □

Like in Hermitian geometry we call 4-dimensional orbits of linear representation of α-quaternion algebra *holomorphic subspaces*. 2-dimensional orbits of the representation are called (*real*) *spinor planes*.

2 Generalized Almost Quaternionic Structures

In this chapter we construct explicitly a fibre bundle of 4-dimensional oriented pseudo-Riemannian manifold on the basis of the well-known notions of self-dual and anti-self-dual forms on the manifold, the type fibre of the bundle being isomorphic to the algebra of generalized quaternions. Such fibre bundle over manifolds of greater dimension is issential in the notion of a generalized almost quaternionic structure introduced and investigated in the chapter. We study in detail the so-called generalized almost quaternionic structures of spinor type, generalized almost quaternionic structures with parallelizable structural bundle being their particular case. The interrelation of the types of generalized almost quaternionic structures is also investigated.

2.1 Features of 4-dimensional pseudo-Riemannian manifolds geometry

Let (M^n, g) be an n-dimensional oriented pseudo-Riemannian manifold, $\mathsf{X}(M)$ be the module of smooth vector fields on M over ring $C^\infty(M)$ of smooth functions on M, $TM = \cup_{p\in M} T_p(M)$ be a tangent bundle over M, $\mathcal{T}(M) = \oplus_{r,s=0}^{\infty} \mathcal{T}_r^s(M)$ be a tensor algebra, and $\Lambda(M) = \oplus_{r=0}^{\infty} \Lambda^r(M)$ be a Grassmann algebra of differential forms of manifold M, $\{E\}$ be a module of smooth sections of fibre bundle (E, M, π) over M. All manifolds, tensor fields and other objects are considered smooth and of class C^∞.

Recall [1] that Hodge operator $* : \Lambda(M) \to \Lambda(M)$ is defined by the sequence of (necessarily unique) operators $* : \Lambda^r(M) \to \Lambda^{n-r}(M); \quad r = 0, \ldots, n$, such that $\omega \wedge (*\vartheta) = \langle \omega, \vartheta \rangle \eta_g$, where $\omega, \vartheta \in \Lambda^r(M)$, η_g is volume form on M, $\langle \cdot, \cdot \rangle$ is metric on $\Lambda^r(M)$ induced by metric $g = \langle \cdot, \cdot \rangle$. We know that $*^2 = (-1)^{r(n-r)+s}\,\mathrm{id}$, where s is the negative inertia index (or, the index) of metric g.

Let, in particular, $n = 4, r = 2$. Then operator $* : \Lambda^2(M) \to \Lambda^2(M)$ is involutory, i.e. $*^2 = \mathrm{id}$, iff s is even, i.e. $s = 0$ or 2 (we take $n - s \geq s$). In this case $\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$, where $\Lambda^+(M)$ and $\Lambda^-(M)$ are eigensubmodules of endomorphism "$*$", corresponding to eigenvalues 1 and (-1), respectively. Their elements are called,respectively, *self-dual* and *anti-self-dual* forms [1].

It is known that giving tensor t of the type (r, s) on smooth manifold is equivalent to giving a system of smooth functions $\{ t^{j_1 \cdots j_s}{}_{i_1 \ldots i_r} \}$ on the space of the frame bundle over the manifold, the functions satisfying the known system of differential equations [28]. The functions are called *components* of tensor t. Denote by $\{ \eta_{ijkl} \}$ components of tensor η_g in a positively oriented frame. Then, if $\{ \omega_{ij} \}$ are components of 2-form $\omega \in \Lambda^2(M)$, then $(*\omega)_{ij} = \frac{1}{2} \eta_{ijkl}\, \omega^{kl}$, i.e. $*\omega_{ij} = \frac{1}{2} g^{km}\, g^{lr}\, \eta_{ijkl}\, \omega_{mr}$, where $\{ g^{ij} \}$ are contravariant components of metric tensor. Thus,

$$\begin{aligned} \omega \in \Lambda^+(M) \qquad &\Longleftrightarrow \qquad \omega_{ij} = \tfrac{1}{2} \eta_{ijkl}\, g^{km}\, g^{lr}\, \omega_{mr}; \\ \omega \in \Lambda^-(M) \qquad &\Longleftrightarrow \qquad \omega_{ij} = -\tfrac{1}{2} \eta_{ijkl}\, g^{km}\, g^{lr}\, \omega_{mr}. \end{aligned} \tag{7}$$

Giving a pseudo-Riemannian structure g on an oriented manifold is equivalent to giving a G-structure having the structural group $G = SO(n, s; \mathbf{R})$. The elements of the G-structure are called *oriented orthonormal frames*. In case $n = 4, s = 0$ or 2 we suppose that vectors of the frames are numerated so that on the space of the G-structure $(g_{ij}) = diag(1, -\alpha, -\alpha, 1)$, where $\alpha = s - 1$. Then correlations (7) have the following form on the space of the G-structure:

$$\omega \in \Lambda^\pm(M) \qquad \Longleftrightarrow \qquad \omega_{\hat{i}\hat{j}} = \pm \varepsilon(i, j)\, \omega_{ij}, \tag{8}$$

where $\varepsilon(i, j) = g_{ii}\, g_{jj}$, $(\hat{i}, \hat{j})$ is a pair complementing the pair (i, j) up to even permutation of indices (1,2,3,4).

Theorem 3. *Giving self-dual form ω on 4-dimensional oriented pseudo-Riemannian manifold (M, g) of index $s = 0$ or 2 is equivalent to giving an endomorphism J of module $\mathsf{X}(M)$ connected with the form ω by the identity*

$$\omega(X, Y) = \langle X, JY \rangle; \qquad X, Y \in \mathsf{X}(M), \tag{9}$$

and having the properties:

$$\begin{gathered} 1.\ \langle JX, Y \rangle + \langle X, JY \rangle = 0; \qquad X, Y \in \mathsf{X}(M); \\ 2.\ J^2 = -\lambda^2 \,\mathrm{id}; \qquad \lambda^2 = -\frac{1}{4}\,\mathrm{tr}\,(J^2) = \frac{1}{2}\|\omega\|^2; \end{gathered} \tag{10}$$

3. *Orientation of manifold M defined by the form $\omega \wedge \omega$ coincides with the orientation fixed on M iff $\lambda \in \mathbf{R} \setminus \{0\}$.*

Proof. Let $\omega \in \Lambda^+(M)$. In view of (8) it implies that on the space of G-structure

$$(\omega_{ij}) = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & z & \alpha y \\ -y & -z & 0 & -\alpha x \\ -z & -\alpha y & \alpha x & 0 \end{pmatrix} \tag{11}$$

Raising index of the form ω we get the matrix of endomorphism J components on the space of G-structure:

$$({J^i}_j) = \begin{pmatrix} 0 & x & y & z \\ \alpha x & 0 & -\alpha z & -y \\ \alpha y & \alpha z & 0 & x \\ -z & -\alpha y & \alpha x & 0 \end{pmatrix} \tag{12}$$

Note that metrix g of index s induces on module $\Lambda^+(M)$ a metric of the same index. Indeed,

$$\|\omega\|^2 = \frac{1}{2}\omega_{jk}\,\omega^{jk} = \frac{1}{2}\sum_{i,j=1}^{4} \epsilon(j,k)(\omega_{jk})^2,$$

where $\epsilon(j,k) = g_{jj}g_{kk}$, and in view of (11), $\|\omega\|^2 = 2(z^2 - \alpha x^2 - \alpha y^2)$. Now, it is clear that (10_1) simply means skew-symmetry of the form ω, (10_2) is checked directly by (12), and (10_3) follows from the definition of Hodge operator.

Inversely, let J be an endomorphism of module $\mathsf{X}(M)$ having properties (10) and ω be the 2-form defined by (9). Then on the space of G-structure

$$(\omega_{ij}) = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & u & v \\ -y & -u & 0 & w \\ -z & -v & -w & 0 \end{pmatrix}; \quad (J^i_j) = \begin{pmatrix} 0 & x & y & z \\ \alpha x & 0 & -\alpha y & \alpha v \\ \alpha y & \alpha x & 0 & -\alpha w \\ -z & -v & -w & 0 \end{pmatrix}$$

Hence $J^2 = -\lambda^2 \,\mathrm{id}$ iff

$$\begin{array}{lll} 1)\ uy = \alpha vz; & 2)\ ux = -\alpha wz; & 3)\ ux = -wy; \\ 4)\ xy = -uw; & 5)\ xz = -\alpha vw; & 6)\ yz = \alpha uv; \end{array}$$

$$\begin{aligned} 7)\ \alpha(x^2+y^2) - z^2 &= \alpha(x^2+v^2) - u^2 = \\ \alpha(y^2+w^2) - u^2 &= \alpha(v^2+w^2) - z^2 = -\lambda^2. \end{aligned}$$

It is easy to see that the system is equivalent to the equations:

$$z = \pm u; \quad y = \pm \alpha v; \quad x = \mp \alpha w; \quad \lambda^2 = z^2 - \alpha x^2 - \alpha y^2. \tag{13}$$

On the other hand, $(\omega\wedge\omega)_{1234} = 2(\omega_{12}\,\omega_{34}+\omega_{13}\,\omega_{42}+\omega_{14}\,\omega_{23}) = 2(xw-yv+zu)$, and in view of (13), $(\omega\wedge\omega)_{1234} = \mp 2(\alpha x^2 + \alpha y^2 - z^2) = 2\lambda^2$. Thus, (10) hold iff $z = u,\ y = \alpha v,\ x = -\alpha w$, i.e. $\omega \in \Lambda^+(M)$. $\square$

Proposition 7. *Let $\omega, \vartheta \in \Lambda^+(M)$; J_1, J_2 be corresponding endomorphisms of module $\mathsf{X}(M)$. Then $\omega \perp \vartheta$ in the induced metric of Grassmann algebra iff $J_1 \circ J_2 + J_2 \circ J_1 = 0$.*

Proof. In view of (10_2), $J^2 = -\frac{1}{2}\|\omega\|^2 \,\mathrm{id} \quad (\omega \in \Lambda^+(M))$. Polarize the equation. Let $\omega, \vartheta \in \Lambda^+(M)$, I and J be their corresponding endomorphisms. Then $(I+J)^2 = -\frac{1}{2}\|\omega+\vartheta\|^2 \,\mathrm{id}$. Opening the brackets and by (10) we have

$$I \circ J + J \circ I = -\langle \omega, \vartheta \rangle \,\mathrm{id},$$

and the assertion being proved immediately follows. $\square$

Identify self-dual forms on M with corresponding endomorphisms of module $\mathsf{X}(M)$ and denote by $\mathcal{S}c\,(M) \subset \mathcal{T}_1^1(M)$ a subbundle of the fibre bundle of scalar endomorhisms on manifold M.

Theorem 4. *Let M be a 4-dimensional oriented pseudo-Riemannian manifold of index 0 or 2. Then with the given identification fibre bundle $\mathsf{O} = \mathcal{S}c\,(M) \oplus \Lambda^+(M)$ over M is a subbundle of fibre bundle $\mathcal{T}_1^1(M)$,its standard fibre being isomorphic to algebra $\mathbf{H}_\alpha$.*

Proof. It follows from the above that in every point $p \in M$ the fibre of bundle O is a subalgebra of algebra $\mathrm{End}\, T_p(M)$. Choose in space $\Lambda_p^+(M)$ an orthogonal basis $\{\,\omega_1, \omega_2, \omega_3\,\}$ whose elements have the norms $\sqrt{-2\alpha}, \sqrt{-2\alpha}, \sqrt{2\alpha}$, respectively. Let J_1 and J_2 be endomorphisms corresponding to the forms ω_1 and ω_2, respectively. In view of Proposition 7 $\ J_1 \circ J_2 + J_2 \circ J_1 = 0$. Besides, by (10_2), $(J_1)^2 = (J_2)^2 = \alpha\,\mathrm{id}$ and thus, the given subalgebra is isomorphic to algebra $\mathbf{H}_\alpha$. $\square$

Remark. In the similar way, it is checked that the anti-self-dual form $\omega \in \Lambda^-(M)$ on the space of G-structure is defined by the matrix of components

$$(\omega_{ij}) = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & -z & -\alpha y \\ -y & z & 0 & \alpha x \\ -z & \alpha y & -\alpha x & 0 \end{pmatrix}$$

Here, the properties of self-dual forms on M formulated in Theorems 3 and 4, as well as in Proposition 7, hold for anti-self-dual forms, too, except (10_3) that is here substituted by:

"Orientation of manifold M, defined by the form $\omega \wedge \omega$, where $\omega \in \Lambda^-(M)$ is inverse to the orientation fixed on M iff $\lambda \in \mathbf{R} \setminus \{0\}$."

Definition 3. *Fibre bundle* O *over 4-dimensional oriented pseudo-Riemannian manifold* M *of index 0 or 2 constructed on the basis of fibre bundle of self-dual (anti-self-dual, respectively) forms on* M *in the above way is called a canonical almost* α*-quaternionic structure of the first* (*second*, respectively) *kind* on the manifold.

2.2 Generalized almost quaternionic structures

Definition 4. We call *an almost* α*-quaternionic* ($AQ_\alpha-$) *structure* on manifold M a subbundle Q of fibre bundle $\mathcal{T}_1^1(M)$, whose standard fibre is algebra $\mathbf{H}_\alpha$. The fibre bundle Q will be called *structural bundle* of AQ_α-structure. The manifold with the fixed AQ_α -structure is called *an almost* α*-quaternionic* ($AQ_\alpha-$) *manifold*.

When $\alpha = -1$, the notion of AQ_α-structure coincides with the classical notion of almost quaternionic (in other terms, almost quaternal) structure [1], [13]. When $\alpha = 1, AQ_\alpha$-structure will be called *an almost antiquaternionic*, or *an almost quaternionic structure of hyperbolic type* (the term is also accepted).

Let Q be an almost α-quaternionic structure on M. Evidently, Q is a 4-dimensional vector fibre bundle metrized by the canonical metric of α-quaternion algebra and associated with the principle fibre bundle $B\mathsf{Q} = (B(\mathsf{Q}), M, \pi, G_\alpha)$, its total space elements being fours $\{\ \mathrm{id}, J_1, J_2, J_3\ \}$ of endomorphisms of tangent space $T_p(M)$ in an arbitrary point $p \in M$ satisfying the correlations $(J_1)^2 = (J_2)^2 = \alpha\,\mathrm{id}, J_1 \circ J_2 + J_2 \circ J_1 = 0, J_3 = J_1 \circ J_2$ and defining oriented orthonormal frames in fibres of fibre bundle Q, and its structure group being Lie group $G_\alpha = SO(2-\alpha, 1+\alpha; \mathbf{R})$. Sections of fibre bundle Q will be called *α-quaternions on M.*

Example 1. Any 4-dimensional oriented pseudo-Riemannian manifold of index $s = 0$ or 2 has, as we have seen, two canonical almost α -quaternionic srtuctures (almost quaternionic, when $s = 0$, and almost antiquaternionic, when $s = 2$).

Example 2. A Riemannian $4n$-dimensional manifold is called (*locally*)*hyper-Kaehler* if its (restricted) holonomy group is contained in $Sp(n)$. The notion of hyper-Kaerher manifold was introduced by E.Calabi [30] who constructed the first non-trivial examples of such manifolds. It is known [1] that a hyper-Kaehler manifold can be characterized as a Riemannian manifold admitting two

anti-commuting complex structures $\{ I, J \}$ parallel in Riemannian connection. Evidently, the algebraic shell of these endomorphisms defines the almost quaternionic structure on the manifold, the structural bundle being parallelizable by endomorphisms $\{ \mathrm{id}, I, J, K \}$, where $K = I \circ J$.

Geometry of hyper-Kaehler manifolds was studied by many authors, for example, M.Berger [31], A.Beauville [19] and others. In particular, M.Berger proved that any hyper-Kaehler manifold is Ricci-flat [31]. A homogeneous hyper-Kaehler manifold is flat, so it is decomposed into the product of tori by Euclidean spaces. It is well-known that a 4-dimensional Riemannian manifold is hyper-Kaehler iff it is a Kaehler and Ricci-flat manifold [1]. At present examples of compact hyper-Kaehler manifolds of any $4n$-dimension are known [19,20]. On the other hand, E.Calabi [30] showed that the space of cotangent bundle of complex projective space $\mathbf{C}P^n$ admits a full hyper-Kaehler metric. Generalizing E.Calabi's constuction, N.Hitchin, A.Karlshede, U.Lindström and M.Roček developed a new factorization method [1] based on symplectic reduction of Marsden and Weinstein [32] allowing to build new examples of non-compact but complete hyper-Kaehler manifolds. With this method, for instance, an example of hyper-Kaehler manifold used for construction of gravitational multi-instanton of Hibbons and Hoking [33] was built.

Example 3. $4n$-dimensional Riemannian manifold is called *quaternionic-Kaehler* if its holonomy group is contained in $Sp(n) \cdot Sp(1)$. It is known [1] that a quaternionic-Kaehler manifold can be characterized as a Riemannian manifold having an almost quaternionic structure, the structural bundle being invariant with respect to parallel translations in Riemannian connection. Since $Sp(1) \cdot Sp(1) = SO(4, \mathbf{R})$, any 4-dimensional oriented Riemannian manifold is quaternionic-Kaehler. Geometry of quaternionic-Kaehler manifolds was also studied by many authors, for example M.Berger [17], J.Wolf [22], D.V.Alekseevskii [23] and other. In particular, M.Berger proved [17] that a quaternionic-Kaehler manifold M^{4n} $(n > 1)$ is Einsteinian, and it is locally hyper-Kaehler iff it is Ricci-flat. J.Wolf [22] received a complete classification of quaternionic-Kaehler symmetric Riemannian manifolds of nonzero Ricci curvature: if such manifold has a positive Ricci curvature it is compact and is one of the following spaces:

1) $Sp(n+1) \,/\, Sp(n) \cdot Sp(1) = \mathbf{H}P^n$;	2) $SU(n+2) \,/\, S(U(n) \cdot U(2))$;
3) $SO(n+4) \,/\, SO(n) \cdot SO(4)$;	4) $G_2 \,/\, SO(4)$;
5) $F_4 \,/\, Sp(3) \cdot Sp(1)$;	6) $E_6 \,/\, SU(6) \cdot Sp(1)$;
7) $E_7 \,/\, Spin(3) \cdot Sp(1)$;	8) $E_8 \,/\, E_7 \cdot Sp(1)$.

If it has a negative Ricci curvature it is non-compact and is one of the following

spaces:

1) $Sp(n,1)\,/\,Sp(n)\cdot Sp(1)$;	2) $SU(n,2)\,/\,S(U(n)\cdot U(2))$;
3) $SO(n,4)\,/\,SO(n)\cdot SO(4)$;	4) $G_2^2\,/\,SO(4)$;
5) $F_4^{-20}\,/\,Sp(3)\cdot Sp(1)$;	6) $E_6^2\,/\,SU(6)\cdot Sp(1)$;
7) $E_7^{-4}\,/\,Spin(3)\cdot Sp(1)$;	8) $E_8^{-24}\,/\,E_7\cdot Sp(1)$.

Moreover, D.V.Alekseevskii [21] proved that any compact homogeneous quaternionic-Kaehler manifold is a Riemannian symmetric space. The question of the existence of non-symmetric quaternionic-Kaehler manifolds was posed by S.Kobayashi and J.Eells [34] and positively solved by D.V.Alekseevskii [23] who received a complete classification of quaternionic-Kaehler manifolds admitting transitive solvable motion group.

Example 4. Almost antiquaternionic structures were studied,up to now, mainly in the case of parallelizable structural bundle. V.F.Kirichenko [35] showed that such structure is canonically defined, for example, on the space of tangent bundle over the manifold of affine connection, as well as on the manifold carrying three-web structure. In the works by M.A.Akivis and A.M.Shelekhov, and many of their pupils, the three-web theory has been developed. In particular, they received a large number of concrete examples of three-webs having a number of remarkable properties, for example, Grassmanian three-webs [26]. In the next chapter we shall see the main notions of tree-webs theory finding their graceful expression in terms of generalized almost quaternionic structure theory. The latter can be regarded, in particular, as a surprising and fruitful three-web theory generalization.

2.3 Almost α-quaternionic connections

Definition 5. An affine connection on AQ_α-manifold is called *an almost quaternionic*, or *AQ_α-connection*, if module Q of structural bundle sections is invariant with respect to all parallel translations generated by the connection.

For example, by definition of quaternionic-Kaehler manifold, Riemannian connection on such manifold is a AQ_α-connection.

Proposition 8. *On AQ_α-manifold M there always exists an AQ_α-connection.*

Proof. Let $\{\,U_a\,\}_{a\in A}$ be the covering of manifold M trivializing bundle Q, and let $\{\,\mathrm{id},(I_a),(J_a),(K_a)\,\}$ be the section of bundle $B\mathbf{Q}$ over U_a. We will see below (Theorem 9) that on manifold U_a there exists connection (∇_a) whose tensors $(I_a),(J_a),(K_a)$ are covariantly constant, in particular, (∇_a) is an AQ_α-connection on U_a. Let $\{\,\varphi_a\,\}_{a\in A}$ be a partitioning of unit dominated by covering $\{\,U_a\,\}_{a\in A}$. Then, evidently, $\nabla=\sum_{a\in A}\varphi_a(\nabla_a)$ is an AQ_α-connection on M. $\square$

Proposition 9. *Any AQ_α-connection ∇ on AQ_α-manifold M induces a metric connection in fibre bundle Q.*

Proof. Let $f \in C^\infty(M), \quad q \in \{\mathsf{Q}\}, \quad X, Y \in \mathsf{X}(M)$. Then

$$\begin{aligned}\nabla_X(fq)Y &= \nabla(fq(Y)) - fq(\nabla_X Y)\\ &= X(f)q(Y) + f\nabla_X(q) + fq(\nabla_X Y) - fq(\nabla_X Y)\\ &= X(f)q(Y) + f\nabla(q)Y,\end{aligned}$$

and in view of arbitrary $Y \in \mathsf{X}(M)$, $\nabla_X(fq) = X(f)q + f\nabla_X(q)$, i.e. ∇ is a linear connection in bundle Q. Further, let $q_1, q_2 \in \{\,\mathsf{Q}\,\}$. Since a AQ_α-connection generates differentiation of algebra $\mathcal{T}(M)$ and the differentiation is permutational with contractions, then

$$\forall X \in \mathsf{X}(M) \Longrightarrow \nabla_X(q_1\, q_2) = \nabla_X(q_1)q_2 + q_1\nabla_X(q_2).$$

Besides, $\nabla_X\,(\overline{q}) = \overline{\nabla_X\,(q)} \quad (q \in \{\,\mathsf{Q}\,\})$. Indeed, in our trivialization

$$q = b\,\mathrm{id} + c(I_a) + d(J_a) + f(K_a); \qquad b, c, d, f \in C^\infty(U_a).$$

Since tensors $(I_a), (J_a), (K_a)$, id are parallel in connection (∇_a),

$$(\nabla_a)_X(q) = X(b)\,\mathrm{id} + X(c)(I_a) + X(d)(J_a) + X(f)(K_a),$$

thus,

$$\nabla_X\,(q) = \sum_{a\in A} \varphi_a\{\,X(b)\,\mathrm{id} + X(c)(I_a) + X(d)(J_a) + X(f)(K_a)\,\}.$$

On the other hand, $\overline{q} = b\,\mathrm{id} - c(I_a) - d(J_a) - f(K_a)$; and hence,

$$\nabla_X(\overline{q}) = \sum_{a\in A} \varphi_a\{\,X(b)\,\mathrm{id} - X(c)(I_a) - X(d)(J_a) - X(f)(K_a)\,\} = \overline{\nabla_X(q)}.$$

It follows that

$$\nabla_X(\overline{q}_1\, q_2) = \overline{\nabla_X(q_1)}q_2 + \overline{q}_1\,\nabla_X(q_2); \quad \nabla_X(\overline{q}_2\, q_1) = \overline{\nabla_X(q_2)}q_1 + \overline{q}_2\,\nabla_X(q_1).$$

Summarizing the equations elementwise and by definition of metric $\langle\cdot,\cdot\rangle$ we get

$$\nabla_X(\langle q_1, q_2\rangle) = \langle\nabla_X\,(q_1), q_2\rangle + \langle q_1, \nabla_X(q_2)\rangle,$$

i.e. ∇ is a metric connection in Q. $\square$

Definition 6. An AQ_α-manifold with fixed AQ_α-connection will be called *calibrated* and the fixed AQ_α-connection will be called *its calibration.*

Further on all considered AQ_α-manifolds will be implied calibrated.

Let M be AQ_α-manifold. In fibre bundle Q we distinguish two natural subbundles – subbundle T of purely imaginary α-quaternions q satisfying the condition $q^2 = \alpha$, or $\alpha(\operatorname{Im} q)^2 = 1$, and subbundle C of α-quaternions q satisfying the condition $\alpha(\operatorname{Im} q)^2 > 0$. We call them, respectively, *twistor* and *conformal bundles* over M associated to structure Q. We call total fibre space of T *a twistor space* over M. The terminology can be explained by the fact that in case M is a 4-dimensional oriented Riemannian manifold, the twistor bundle over M with respect to canonical almost quaternionic structure coincides with the classical twistor Penrose bundle [1]. In case M is an arbitrary quaternionic manifold, that is an almost quaternionic manifold, admitting almost quaternionic torsion-free connection [5], the twistor bundle over M coincides with the twistor bundle introduce by S.Salamon [5] and L.Beŕard-Bergery [6].

Sections of fibre bundle T will be called *twistors* on M. Evidently, any twistor is either an almost complex structure or a structure of almost product on M when $\alpha = -1$ or 1, respectively. Note that in view of Proposition 9 an AQ_α-connection on M induces a connection in conformal bundle C.

2.4 Generalized almost quaternionic structure of spinor type

Definition 7. An almost α-quaternionic structure on manifold M is called *a structure of spinor type* if M admits a twistor.

Definition 8. An almost α-quaternionic structure on M is called *an πAQ_α-structure* if M admits a pair of anticommuting twistors.

The above condition is, evidently, equivalent to parallelizability of structural bundle: if $\{ I, J \}$ is a pair of anticommuting twistors on M, then endomorphisms $\{ \mathrm{id}, I, J, K \}$, where $K = I \circ J$, define the parallelizma of structural bundle.

For example, any hyper-Kaehler manifold, as well as the manifold carrying a three-web structure is a πAQ_α-manifold and, certainly, a AQ_α-manifold of spinor type. On the other hand, by [1], quaternionic projective space $\mathbf{H}P^n$, being a quaternionic-Kaehler manifold, does not admit an almost complex structure and, thus, is not an AQ_α-manifold of spinor type. By similar reason, 4-dimentional sphere S^4, being a quaternionic-Kaehler manifold, is not an AQ_α-manifold of spinor type.

Theorem 5. *The space of conformal bundle over AQ_α-manifold M admits a natural structure of an AQ_α-manifold of spinor type.*

Proof. Let ϑ be the connection form in fibre bundle C induced by calibration $\nabla, q \in \mathsf{C}, q_1 \in \mathsf{C}_{\pi(q)}(M), X \in T_q(\mathsf{G})$. Assume

$$Q_1(X) = \tau^{-1} \circ L_{q_1} \circ \tau \circ \vartheta(X) + \pi_*^{-1} \circ q_1 \circ \pi_*(X), \tag{14}$$

where π_*^{-1} is the operator of horizontal lifting from $T_{\pi(q)}(M)$ into horizontal distribution of connection ϑ, τ is the operator of identification of vertical subspace at point q with fibre $\mathsf{C}_{\pi(q)}(M), L_{q_1}$ is the left shift in the fibre on α-quaternion q_1. Evidently, the family of operators constructed in this way defines an AQ_α-structure on the space of bundle C, and endomorphism J generated by operators $J_q = |\mathrm{Im}\, q|^{-1} \cdot \mathrm{Im}\, q$ at every point $q \in \mathsf{C}$ defines the twistor on space C. □

Denote the constructed AQ_α-structure on manifold C by Q_c and call it *conformal lifting* of the initial AQ_α-structure Q Theorem 5 reduces geometry of an arbitrary AQ_α-structure Q to the geometry of a certain AQ_α-structure of spinor type — conformal lifting of structure Q.

Definition 9. An almost α-quaternionic structure Q on manifold M is called *integrable* if there exists atlas $\{ U_a, \varphi_a \}_{a\in A}$ of manifold M trivializing fibre bundle $B(\mathsf{Q})$, the sections $\{ I_a, J_a \}_{a\in A}$ defining trivialization over $U_a, \quad a \in A$ being given in the natural basis by constant matrices.

¿From (4) immediately follows

Proposition 10. *Integrability of AQ_α-structure is equivalent to integrability of its conformal lifting.* □

Let Q be an almost α-quaternionic structure of spinor type on manifold M, J be a fixed twistor on M. In fibre bundle Q there is naturally introduced the structure of 2-dimensional Hermitian vector fibre bundle $\mathcal{S}$ whose standard fibre is α-spinvector space $\mathcal{S}_0$ and Hermitian metric is a Hermitian form $\langle\langle X, Y\rangle\rangle = \langle X, Y\rangle + j\langle X, J^3Y\rangle$. We call the fibre bundle *the bundle of α-spinvectors over M.* It is associated to the principal bundle $B\mathcal{S} = (B(\mathcal{S}), M, \pi, SCo(2, \mathbf{K}_\alpha))$ whose structural group is Lie group $\mathbf{H}_\alpha^* = SCo(2, \mathbf{K}_\alpha) = SU(2, \mathbf{K}_\alpha) \times \mathbf{R}^+ = Sp_\alpha(1) \times \mathbf{R}^+$ of non-isotropic really normalized α-quaternions. The group is regarded as one of special conformal transformations of α-spinvector space $\mathcal{S}_0$. The elements of total fiber space $B(\mathcal{S})$ are orthogonal frames of the fibres of fiber bundle $\mathcal{S}$, $\mathbf{R}$-homothetic to spinframes.

Proposition 11. *There exists a one-to-one corespondence between manifolds $B(\mathsf{Q})$ and $B(\mathcal{S})/\mathbf{R}^*$, where $\mathbf{R}^*$ is a multiplicative group of real numbers acting on the fibres of fibre bundle as a group of homotheties.*

Proof. Let $p = (m, \mathrm{id}, J_1, J_2, J_3) \in B(\mathsf{Q})$ be a positively oriented orthonormalized frame of space Q_m. By Theorem 1 and Proposition 5 it has a corresponding class $rU(1, \mathbf{K}_\alpha)$ of orthonormalized frames of space $\mathcal{S}_m$, where $(J_k) = \sigma_k, \quad k = 1, 2, 3$. Let $\tilde{r} \in B(\mathcal{S}) \cap rU(1, \mathbf{K}_\alpha)$. Then, $\tilde{r} = rg$, where $g = diag(e^{ia}, e^{ia})$, $\det g = 1$. Then, $e^{2ia} = 1$, that is $a = \pi k, g = \pm I$. Keeping in mind that each orthonormalized frame r has a corresponding class $r\mathbf{R}^+$ of orthogonal frames with the same determining property, belonging to the orbit of frame r with respect to the structural group of fibre bundle $B\mathcal{S}$, we get $\{ r \in B(\mathcal{S}) \mid (J_k) = \sigma_k, k = 1, 2, 3 \} = r\mathbf{R}^*$. The juxtaposition $p \to r\mathbf{R}^*$ gives the desired correspondence. □

Proposition 12. *There exists an innerly defined* Ad*-homomorphism* $S: B\mathcal{S} \to B\mathsf{Q}$ *of principal fibre bundles juxtaposing to the orthogonal frame* $p = (m, \varepsilon_1, \varepsilon_2) \in B(\mathcal{S})$ *a positively oriented orthonormalized frame* $(m, \mathrm{id}, J_1, J_2, J_3) \in B(\mathsf{Q})$*, such that* $(J_k) = \sigma_k; \quad k = 1, 2, 3.$

Proof. Let $p = (m, \varepsilon_1, \varepsilon_2) \in B(\mathcal{S})$. Then $S(pg) = (m, \mathrm{id}, J_1, J_2, J_3) \mid (J_k)_{pg} = \sigma_k; \quad k = 1, 2, 3$. Note that $(J_k)_{pg} = g^{-1}(J_k)_p\, g$, i.e. $(J_k)_p = g\sigma_k\, g^{-1}$. On the other hand, $S(p) = (m, \mathrm{id}, \tilde{J}_1, \tilde{J}_2, \tilde{J}_3) \mid (\tilde{J}_k)_p = \sigma_k; \quad k = 1, 2, 3; \quad S(p)\,\mathrm{Ad}\,(g) = (m, \mathrm{id}, \hat{J}_1, \hat{J}_2, \hat{J}_3) \mid (\hat{J}_k)_p = (\,\mathrm{Ad}\,(g)(\tilde{J}_k))_p = g(\tilde{J}_k)_p g^{-1}$. By comparison, we get $[\hat{J}_k]_p = [J_k]_p$, i.e. $S(pg) = S(p)\,\mathrm{Ad}\,(g)$. $\square$

Lemma 1. *Let* $s : G_1 \to G_2$ *be a Lie group homomorphism. Then* $\forall g \in G_1 \Longrightarrow \mathrm{Ad}\,(sg) \circ s_* = s_* \circ \mathrm{Ad}\,(g)$*, where* s_* *is differential of mapping* s *in the Lie group unit.*

Proof. Let A_g be the inner Lie group automorphism generated by element g. Then $(A_{s(g)} \circ s)h = s(g)s(h)s(g^{-1}) = s(ghg^{-1}) = (s \circ \mathrm{Ad}\,)h \quad (h \in G_1)$. So, $A_{s(g)} \circ s = s \circ A_g$. We complete the proof passing onto mapping differentials. $\square$

Proposition 13. *Any connection in the principle fibre bundle* $(B(\mathsf{Q}), M, \pi, G_\alpha)$ *induces a connection in the principle fibre bundle* $(B(\mathcal{S}), M, \pi, SCo(2, \mathbf{K}_\alpha))$.

Proof. Let Θ be a connection form on $B(\mathsf{Q})$. We define the form $\vartheta = s_*(\Theta)$ on $B(\mathsf{S})$. We have:

$$\begin{aligned} \vartheta_{pg} &= s_*^{-1}\Theta_{S(p)g}\, s_* = s_*^{-1}\,\mathrm{Ad}\,(s(g^{-1}))\Theta_{S(p)}\, s_* = \\ &= \mathrm{Ad}\,(g^{-1})s_*^{-1}\Theta_{S(p)}s_* = \mathrm{Ad}\,(g^{-1})\vartheta_p. \end{aligned}$$

Thus, ϑ is a connection form on $B(\mathcal{S})$. $\square$

We call this connection *a spinor representation of initial connection*.

Let M be an almost α-quaternionic manifold of spinor type. By Proposition 9, its calibration induces a metric connection in fibre bundle $B\mathsf{Q}$. Let ϑ be the form of spinor representation of the connection, $\mathcal{H}$ and $\mathcal{V}$ be horizontal and vertical distributions on $B(\mathcal{S})$, respectively. Fix $p \in B(\mathcal{S})$. Let $S(p) = (m, \mathrm{id}, J_1, J_2, J_3)$ be its corresponding positively oriented orthonormalized frame of space Q_m. Define endomorphisms $\mathcal{J}_k \in \mathcal{T}_1^1(B(\mathcal{S}))$ by the formulas

$$(\mathcal{J}_k)_p = \vartheta^{-1} \circ [J_k] \circ \vartheta + \pi_*^{-1} \circ J_k \circ \pi_*; \qquad k = 1, 2, 3; \tag{15}$$

where π_*^{-1} is the operator of horizontal lifting from $T_m(M)$ to $\mathcal{H}_p$, ϑ^{-1} is the identification operator of Lie algebra $\mathsf{sp}_\alpha(1) \oplus t^1$ with $\mathcal{V}_p$. Evidently, the pair $\{\,\mathcal{J}_1, \mathcal{J}_2\,\}$ defines an almost α-quaternionic structure with parallelizable structural bundle, that is a πAQ_α-structure, on manifold $B(\mathcal{S})$.

Denote $\mathcal{J}_k^V = \vartheta^{-1} \circ [J_k] \circ \vartheta$, $\mathcal{J}_k^H = \pi_*^{-1} \circ J_k \circ \pi_*$. Call the endomorphisms *vertical* and *horisontal components of the structure* $\{ \mathcal{J}_1, \mathcal{J}_2 \}$, respectively. Evidently, distributions $\mathcal{V}$ and $\mathcal{H}$ are invariant with respect to endomorphisms $\mathcal{J}_k^V$ and $\mathcal{J}_k^H$, respectively. Note that $\mathcal{J}_k^H = J_k^H$ is just horizontal lifting of twistor $J_k \quad (k = 1, 2)$. In this notation (15) will have the form $\mathcal{J}_k = \mathcal{J}_k^V + \mathcal{J}_k^H$. Consider the dependence of the endomorphisms on the right action of the structural group in fibre space. Let $g \in \mathbf{H}_\alpha^\times$. Then

$$
\begin{aligned}
(\mathcal{J}_k^V)_{pg} &= \vartheta_{pg}^{-1}[J_k]_{pg}\vartheta_{pg} \\
&= (\operatorname{Ad}(g^{-1})\vartheta_p)^{-1}[(J_k)_{pg}] \operatorname{Ad}(g^{-1})\vartheta_p \\
&= \vartheta_p^{-1}(\operatorname{Ad}(g^{-1})^{-1})[(J_k)_{pg}] \operatorname{Ad}(g^{-1})\vartheta_p \\
&= \vartheta_p^{-1} g \circ [(J_k)_{pg}] \circ g^{-1}\vartheta_p \\
&= \vartheta_p^{-1}[(J_k)_p]\vartheta_p = (\mathcal{J}_k^V)_p; \\
(\mathcal{J}_k^H)_{pg} &= (J_k^H)_{pg} = \pi_*^{-1}(s(g)J_k)\pi_* = (s(g)J_k)_p^H.
\end{aligned}
$$

Thus, we have proved

Proposition 14. *With the given notation*

$$
1)\, (\mathcal{J}_k^V)_{pg} = (\mathcal{J}_k^V)_p; \quad 2)\, (\mathcal{J}_k^H)_{pg} = (s(g)J_k)_p^H. \quad \square
$$

Summarizing the above we see that on total space $B(\mathcal{S})$ of the principle fibre bundle $B\mathcal{S}$ over an almost α-quaternionic manifold of spinor type there is canonically induced a structure of an almost α-quaternionic manifold with parallelizable structural bundle generated by endomorphisms $\{ \mathcal{J}_1, \mathcal{J}_2 \}$.

Definition 10. Manifold $B(\mathsf{S})$, equipped with a πAQ_α-srtucture generated by endomorphisms $\{ \mathcal{J}_1, \mathcal{J}_2 \}$ is called *a covering* $\mathcal{S}$*-space* of AQ_α-manifold M of spinor type, and a πAQ_α-structure generated by the endomorphisms is called *a covering* or *canonical* πAQ_α*-structure*.

By Proposition 14 we get that the natural projection π of fibre space $B\mathcal{S}$ generates "projecting" of endomorphisms $\{ (\mathcal{J}_k)_p \}$, $p \in B(\mathcal{S})$, of the covering πAQ_α-structure into tensor fibre space of the type (1,1) on M, whose image coincides with the space of structural bundle Q of the initial AQ_α-structure. Thus, the covering πAQ_α-structure is defined by the initial AQ_α-structure and, in its turn, defines it. We get

Theorem 6. *Giving of almost α-quaternionic structure of spinor type on manifold is equivalent to giving of its covering πAQ_α-structure on the covering $\mathcal{S}$-space.* $\square$

2.5 Spintensor algebra of generalized almost quaternionic manifold of spinor type

Let M be an almost α-quaternionic manifold of spinor type, $\mathcal{S}$ be a bundle of α-spinvectors over M. By means of anti-automorphism τ of complex conjugation in module $\mathcal{S}_0$ (the standard fibre of the bundle) we can introduce another structure of $\mathbf{K}_\alpha$-module, assuming $z \circ X = \tau(z)\,X$. Denote module $\mathcal{S}_0$ equipped with such structure by $\overline{\mathcal{S}}_0$ and call it *a module conjugated to module* $\mathcal{S}_0$. We also introduce module $\mathcal{S}_0^*$, dual to module $\mathcal{S}_0$, and module $\overline{\mathcal{S}}_0^*$, dual to module $\overline{\mathcal{S}}_0$. The section module of direct sum $\mathcal{S} \oplus \overline{\mathcal{S}} \oplus \mathcal{S}^* \oplus \overline{\mathcal{S}}^*$ of corresponding bundles generates in a standard way a tensor algebra $\mathcal{S}(M)$ that we call *a spinors algebra of manifold M*, its elements being called *α-spinor on M.* The algebra has a natural 4-graduation: α-spinor of the type $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be regarded as a linear mapping of the module

$$(\otimes^a \{\, \mathcal{S} \,\}) \otimes (\otimes^b \{\, \overline{\mathcal{S}} \,\}) \otimes (\otimes^c \{\, \mathcal{S}^* \,\}) \otimes (\otimes^d \{\, \overline{\mathcal{S}}^* \,\})$$

in $C^\infty(M)$. Finally, tensor product $\mathcal{S}(M) \otimes \mathcal{T}(M)$ is called *a spintensor algebra of manifold M* and denote by $\mathcal{ST}(M)$ its elements being called *α-spintensors on M*.

Let ϑ be the form of spinor representation of metric connection in fibre bundle $B\mathsf{Q}$ induced by calibration of manifold M. We construct the mapping f of vertical right-invariant vector field module on manifold $B(\mathsf{Q})$ onto module of α-quaternions on M, assuming $(f(X))_{\pi}(p) = S(p)\,\vartheta_p(X_p)$, where $S(p) : \mathbf{H}_\alpha \to \mathsf{Q}_{\pi(p)}$ is a canonical mapping defined by frame $S(p)$. We show the correctness of the definition in the sense of independence on the choice of frame $p \in B(\mathcal{S})$. Let $\pi(p) = \pi(p')$; $p' = pg$, $g \in SCo(2, \mathbf{K}_\alpha)$. Then

$$\begin{aligned} S(p')\,\vartheta_{p'}(X_{p'}) &= S(pg)\,\vartheta_{pg}(X_{pg}) \\ &= S(p)\,\mathrm{Ad}\,(g)\,\mathrm{Ad}\,(g^{-1})\,\vartheta_p(X_p) \\ &= S(p)\,\vartheta_p(X_p) = (f(X))_{\pi(p)}, \end{aligned}$$

i.e. the definition is correct.

Thus, the mapping f identifies vertical right-invariant vector fields on manifold $B(\mathcal{S})$ with α-quaternions on manifold M. On the other hand, if X is an arbitrary right-invariant vector field on $B(\mathcal{S})$, then $X = X^V + X^H$ and in view of canonical identification of right-invariant horizontal vector fields on $B(\mathcal{S})$ with elements from $\mathsf{X}(M)$ we get:

Theorem 7. *Module $\mathsf{X}^R(B(\mathcal{S}))$ of right-invariant vector fields on the covering $\mathcal{S}$-space of AQ_α-manifold of spinor type is canonically identified with the direct sum of modules $\{\, \mathsf{Q} \,\} \oplus \mathsf{X}(M)$.* □

Note that module $\{\, \mathsf{Q} \,\}$ is naturally identified with a submodule of spinor algebra of manifold M generated of Hermitian form $\langle\langle \cdot, \cdot \rangle\rangle$ and the submodule

skew-Hermitian forms on $\{\mathsf{Q}\}$. Indeed, let q be an arbitrary α-quaternion on M, $q = (\mathrm{Re}\, q)\,\mathrm{id} + \mathrm{Im}\, q$. Assume

$$\begin{aligned} Q(X,Y) &= \langle\langle X, q(Y)\rangle\rangle \\ &= \langle\langle X, (\mathrm{Re}\, q)Y\rangle\rangle + \langle\langle X, (\mathrm{Im}\, q)Y\rangle\rangle \\ &= \langle\langle X, Y\rangle\rangle\, \mathrm{Re}\, q + \langle\langle X, (\mathrm{Im}\, q)Y\rangle\rangle; \qquad X, Y \in \{\mathcal{S}\} \end{aligned}$$

Evidently,

$$\begin{aligned} Q(X,Y) &= \langle\langle X, (\mathrm{Im}\, q)Y\rangle\rangle = \mathrm{Re}\,(\overline{X}\cdot \mathrm{Im}\, q \cdot Y) = \mathrm{Re}\,(\overline{\overline{Y}\cdot \overline{\mathrm{Im}\, q}\cdot X}) \\ &= \overline{\mathrm{Re}\,(\overline{Y}\cdot \overline{\mathrm{Im}\, q}\cdot X)} = -\overline{\mathrm{Re}\,\overline{Y}\cdot \mathrm{Im}\, q\cdot X)} = -\overline{\langle\langle Y, (\mathrm{Im}\, q)X\rangle\rangle} \\ &= -\overline{Q(Y,X)}, \end{aligned}$$

i.e. Q is a skew-Hermitian form. In view of this identification the direct sum $\{\mathsf{Q}\}\otimes\mathsf{X}(M)$ generates a subalgebra of spintensor algebra of manifold M. Thus, by Theorem 7, we get

Theorem 8. *There exists a canonical non-graduated monomorphism of tensor algebra of right-invariant tensor fields on the covering $\mathcal{S}$-space of an AQ_α-manifold of spinor type into the spintensor algebra of the manifold.* □

The monomorphism juxtaposes the right-invariant tensor $t \in \mathcal{T}^R(B(\mathcal{S}))$ to a defined set of spintensors $\{(t_k)\}$ on M, where $(t_k)(X_1,\dots,X_m) = t(\psi X_1,\dots$ $\dots,\psi X_m)$, where ψX_j is either $f^{-1}(X_j) \in \mathcal{V}$ or $\mathcal{V}^*$, either $\pi_*^{-1}(X_j) \in \mathcal{H}$ or $\mathcal{H}^*$, $\quad j = 1,\dots,m$, depending on whether there is 0 or 1 in the j-th place of the binary representation of k. The set $\{(t_k)\}$ of spintensors defining the right-invariant tensor t is called *the spectre* of tensor t.

3 Generalized Almost Quaternionic Manifolds with Parallelizable Structural Bundle

The notion of an almost quaternionic structure was, evidently, first considered by P.Libermann [8] and further studied by M.Obata [10], S.Ishihara [11] and many other authors. The almost quaternionic structures studied in mentioned papers were defined by giving of two anticommuting almost complex structures and, thus, had a parallelizable structural bundle. The same feature was characteristic of almost antiquaternionic structures generated by geometry of tangent bundle, three-web structure, as well as by almost Hermitian structure of hyperbolic type on Riemannian manifold [35]. The present chapter is devoted to the study of the structures from a single view point as πAQ_α-structures. This class of AQ_α-structures is of importance since (as we saw in the previous chapter) the study of any AQ_α-structure is generally reduced to the study of its covering πAQ_α-structure. The main result of the present chapter is having proved that

πAQ_α-structure on a manifold innerly generates a unique AQ_α-connection of a special type that we call *canonical the connection*, in particular, generalizes the Chern connection widely used in three-web theory [36]. A formula for global definition of canonical connection is received. Criteria of integrability of πAQ_α-structure and its structural endomorphisms are found in terms of torsion and curvature tensors of canonical connection. We also introduce the notions of isoclinic distribution and isoclinic πAQ_α-structure, as well as isoclinic-geodesic πAQ_α-structure, that generalizes the corresponding notions of three-web theory [37]. Finally, we find a criterion of semi-holonomic πAQ_α-structure being isoclinic generalizing the well-known criterion of M.A.Akivis in three-web theory [37].

3.1 Canonical connection of πAQ_α-structure

Recall that an almost α-quaternionic structure with parallelizable structural bundle is uniquely defined by a pair of anticommuting twistors. Consider them fixed and assume the following definition equivalent to Definition 8:

Definition 11. A pair $\{ I, J \}$ of endomorphisms of module $\mathsf{X}(M)$ for which

$$1)\, I^2 = J^2 = \alpha\, \mathrm{id}; \quad 2)\, I \circ J + J \circ I = 0 \qquad (\alpha = \pm 1),$$

is called *πAQ_α-structure on manifold M.*

The mentioned definitions are equivalent since the algebraic shell of endomorphisms I and J defines subbundle Q of a fibre bundle of tensors of the type (1,1) on M with standard fibre $\mathbf{H}_\alpha$, that is an AQ_α-structure on M, I and J being a pair of anticommuting twistors with respect to the structure. We call I, J and $K = I \circ J$ *structural endomorphisms* of πAQ_α-structure.

Theorem 9. *Any πAQ_α-manifold $\{ M, I, J \}$ admits a unique connection ∇ having the following properties:*

$$1)\, \nabla I = 0; \quad 2)\, \nabla J = 0; \quad 3)\, S(IX, Y) - S(X, IY) = 0; \tag{16}$$

where $S(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ *is torsion tensor of the connection ∇, $[\cdot,\cdot]$ are Lie brackets, $X, Y \in \mathsf{X}(M)$.*

Proof. In $\mathbf{K}_\alpha \otimes C^\infty(M)$-module $\mathbf{K}_\alpha \otimes \mathsf{X}(M)$ there are naturally defined four pair-wise conjugate submodules — the eigendistributions $D_I^i, D_I^{-i}, D_J^i, D_J^{-i}$ of structural endomorphisms with eigenvalues $i, -i, i, -i$, respectively. Besides, there are naturally defined endomorphisms $V = \frac{1}{2}(\mathrm{id} + iI^3)$ and $H = \frac{1}{2}(\mathrm{id} - iI^3)$ that are projectors complementary to each other on distributions D_I^i and D_I^{-i}, respectively. Let $X, Y \in \mathsf{X}(M)$, $X = VX + HX$, $Y = VY + HY$. Denote $VX = a$, $VY = b$, $HX = u$, $HY = v$. Then $X = a + u$, $Y = b + v$. Assume that the desired connection exists. Then

$$\nabla_X (IY) = \nabla_X (I)Y + I\nabla_X Y = I\nabla_X Y; \tag{17}$$

$$\nabla_X (JY) = \nabla_X (J)Y + J\nabla_X Y = J\nabla_X Y. \tag{18}$$

Further, $iS(a,u) = S(ia,u) = S(Ia,u) = S(a,Iu) = S(a,-iu) = iS(a,u)$, hence $S(a,u) = 0$, and thus, $[a,u] = \nabla_a u - \nabla_u a - S(a,u) = \nabla_a u - \nabla_u a$. But by (17), $\nabla_a u = \nabla_a (HX) = H\nabla_a X \in D_I^{-i}$, $\nabla_u a = \nabla_u (VY) = V\nabla_u X \in D_I^{i}$. Then,

$$\nabla_a u = H[a,u], \quad \nabla_u a = -V[a,u] = V[u,a].$$

But then in view of (18),

$$\begin{aligned} \nabla_a b &= \nabla_a (J\tilde{u}) = J\,\nabla_a \tilde{u} = JH[a,\tilde{u}] = \alpha JH[a,Jb] \\ &= \alpha VJ[a,Jb] = VJ^3\,[a,Jb]; \\ \nabla_u v &= \nabla_u (J\tilde{b}) = J\nabla_u \tilde{b} = JV[u,\tilde{b}] = \alpha JV[u,Jv] \\ &= \alpha HJ[U,jV] = HJ^3[u,Jv]. \end{aligned}$$

Therefore,

$$\nabla_X Y = \nabla_a b + \nabla_a v + \nabla_u b + \nabla_u v = VJ^3[a,Jb] + H[a,v] + V[u,b] + HJ^3[u,Jv],$$

or in view of the notations,

$$\nabla_X Y = V\{\,[HX,VY] + J^3\,[VX,JVY]\,\} + H\{\,[VX,HY] + J^3\,[HX,JHY]. \quad (19)$$

In view of definitions V and H, (19) we can write explicitly the following:

$$\begin{aligned} \nabla_X Y = \frac{1}{4}\{\,[X,Y] - \alpha[IX,IY] + \alpha J[X,JY] - J[IX,IJY] - \\ -\alpha I[IX,Y] + \alpha I[X,IY] - IJ[X,IJY] + IJ[IX,JY]\,\}. \quad (20) \end{aligned}$$

Inversely, immediate checking show that ∇ is affine connection on M having the required properties. $\square$

We call the constructed connection *a canonical connection of a πAQ_α-structure*. Identities 16_1 and (16_2) show that when $\alpha = -1$ connection ∇ is almost quaternionic in the sense of M.Obata [10]. On the other hand, considering the almost antiquaternionic structure joined to a three-web [35] easily shows that invariant connection joined to a three-web [36] later called *Chern connection* has the properties of the canonical connection of adjointed πAQ_α-structure and, thus, coincides with it.

Theorem 10. *πAQ_α-structure is integrable iff torsion and curvature tensors of its canonical connection are zero.*

Proof. Integrability of πAQ_α-structure $\{\,I,J\,\}$ means the existence of an atlas of manifold M^n carrying the structure, tensors I and J having constant coordinates in every map of the atlas. Fix map (U,φ) of such atlas with local coordinates $(x^1,\ldots,x^n)$ and let $\{\,e_k\,\}$ be the natural map basis, $e_k = \frac{\partial}{\partial x^k}$, $k = 1,\ldots,n = \dim M$. Here $\varphi : U \to V \subset R^n$ is the mapping of the map. Any vector $X \in \mathbf{R}^n$ generates an one-parametric group of diffeomorphisms $\{\,T_t\,\}$, $T_t(Y) = Y + tX$ of the space $\mathbf{R}^n$ and, respectively, a

local one-parameter group of diffeomorphisms of domain V that we denote by the same symbol. Evidently, $T_t(\tilde{e}_k) = (T_t)_* \, \tilde{e}_k = \tilde{e}_k$, where $\tilde{e}_k = \varphi_*(e_k)$, f_* is differential of mapping f. Accordingly, in domain U there acts a local one-parametric group of diffeomorphisms $\{ F_t \}$ where $F_t = \varphi^{-1} \circ T_t \circ \varphi$. Evidently, $(F_t)_* \, e_k = (\varphi^{-1} \circ T_t)_* \, \tilde{e}_k = (\varphi^{-1})_* \, \tilde{e}_k = e_k$. Thus, $\mathcal{L}_{(\varphi^{-1})_* \, X} (e_k) = 0$, where $\mathcal{L}$ is the operator of Lie differentiation. In particular, $\mathcal{L}_{e_i}(e_j) = 0$, that is

$$[e_i, e_j] = 0; \qquad i, j = 1, \dots, n. \tag{21}$$

Without the loss of generality it can be considered that the given basis has the form $\{ e_1, \dots, e_m, Ie_1, \dots, Ie_m \}$. Introduce the complex basis $\{ \varepsilon_1, \dots, \varepsilon_m, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{m}} \}$, where $\varepsilon_a = \frac{1}{2}(\,\mathrm{id} + iI^3)e_a$ and $\varepsilon_{\hat{a}} = \frac{1}{2}(\,\mathrm{id} - iI^3)e_a$, $\quad a = 1, \dots, m$. Evidently, $\{ \varepsilon_a \}$ is the basis of distribution $D_I^i|U$. From (21) we have that $[\varepsilon_a, \varepsilon_{\hat{b}}] = 0$. But then $\nabla_{\varepsilon_a} \varepsilon_{\hat{b}} = H[\varepsilon_a, \varepsilon_{\hat{b}}] = 0$, hence in the above notations $\nabla_a \, u = \nabla_{a^b \varepsilon_b}(u^{\hat{c}} \, \varepsilon_{\hat{c}}) = a^b \, \nabla_{\varepsilon_b}(u^{\hat{c}} \, \varepsilon_{\hat{c}}) = a^b \, \varepsilon_b \, (u^{\hat{c}})\varepsilon_{\hat{c}} = a(u^{\hat{c}})\varepsilon_{\hat{c}}$. Similarly, $\nabla_u \, a = u(a^c)\varepsilon_c$. Further, $\nabla_{\varepsilon_a} \varepsilon_b = VJ^3[\varepsilon_a, J\varepsilon_b] = 0$, $\nabla_{\varepsilon_{\hat{a}}} \varepsilon_{\hat{b}} = HJ^3[\varepsilon_{\hat{a}}, J\varepsilon_{\hat{b}}] = 0$, hence, $\nabla_a \, b = a(b^c)\varepsilon_c$; $\nabla_u \, v = u(v^{\hat{c}})\varepsilon_{\hat{c}}$, and thus, $\nabla_X \, Y = X(Y^a)\varepsilon_a + X(Y^{\hat{a}})\varepsilon_{\hat{a}} = X(Y^k)\varepsilon_k$, $\quad k = 1, \dots, n$. Consequently,

$$\begin{aligned}
S(X,Y) &= \nabla_X \, Y - \nabla_Y \, X - [X,Y] = X(Y^i)\varepsilon_i - Y(X^i)\varepsilon_i - [X,Y]^i \varepsilon_i = 0; \\
R(X,Y)Z &= \nabla_X \nabla_Y \, Z - \nabla_Y \nabla_X \, Z - \nabla_{[X,Y]} \, Z \\
&= X(Y(Z^k))\varepsilon_k - Y(X(Z^k))\varepsilon_k - [X,Y](Z^k)\varepsilon_k = 0.
\end{aligned}$$

Inversely, if torsion and curvature tensors of canonical connection are zero, then there exists an atlas where Christoffel connection coeffitients are equal to zero, and then by (16) in the corresponding local coordinate system the partial derivatives of tensors I and J coordinates are equal to zero, i.e. the tensors have constant coordinates in the corresponding local maps. $\square$

3.2 Fundamental distributions and integrability

Definition 12. Let (M, I, J) be a πAQ_α-manifold. On it there are innerly defined eigendistributions $D_I^\lambda, D_J^\mu, D_K^\nu$ corresponding to eigenvalues λ, μ, ν of endomorphisms $I, J, K = I \circ J$ respectively. We call the distributions *fundamental.* The distributions D_I^λ are called *the principal fundamental distributions.*

Evidently, λ and μ are equal to $\pm i$ or ± 1, $\nu = \pm i$. The question naturally arises about the conditions of the distributions being involutive. It finds its graceful solution in terms of canonical connection.

Let ∇ be canonical connection of πAQ_α-structure, S be its torsion tensor. We introduce in module $\mathsf{X}(M)$ (as well as in tangent space $T_p(M)$, $p \in M$) a structure of anticommutative algebra with composition operation $X * Y = S(X, Y)$; $\quad X, Y \in \mathsf{X}(M)$, and extend it by linearity on module $\mathbf{H}_\alpha \otimes \mathsf{X}(M)$

(resp., $\mathbf{H}_\alpha \otimes T_p(M)$). We call the algebra *adjoint* and denote it by V (resp., V_p). By (16_3),

$$IX * Y = X * IY; \qquad X, Y \in \mathsf{V}.$$

Theorem 11. *Fundamental distribution of πAQ_α-structure is involutive iff it is a subalgebra of the adjoint algebra. Here the principal fundamental distribution is involutive iff it is the ideal of the adjoint algebra.*

Proof. Let F be one of structural endomorphisms I, J or K, λ be its eigenvalue, D_F^λ be the corresponding fundamental distribution, $D_F^{-\lambda}$ be the complementary fundamental distribution, $\pi^+ = \frac{1}{2}(\,\mathrm{id} + \lambda F^3)$ be projector on D_F^λ, $\pi^- = \frac{1}{2}(\,\mathrm{id} - \lambda F^3)$ be projector on $D_F^{-\lambda}$. Assume that D_F^λ is involutive, i.e. $\forall X, Y \in D_F^\lambda \Longrightarrow [X, Y] \in D_F^\lambda$. This condition can be written in the form $\pi^-[\pi^+ X, \pi^+ Y] = 0; \quad X, Y \in \mathsf{X}(M)$. In view of definition of endomorphisms π^+ and π^- it can be rewritten in the form

$$\alpha N_F(X, Y) - \lambda F \circ N_F(X, Y) = 0; \qquad X, Y \in \mathsf{X}(M), \tag{22}$$

where $N_F(X, Y) = \alpha[X, Y] + [FX, FY] - F[FX, Y] - F[X, FY]$ is Nijenhuis tensor of the endomorphism F. Evidently, equality (22) can be written in the form

$$\lambda F^3 \circ N_F(X, Y) = N_F(X, Y), \quad \text{i.e.} \quad (\pi^-) \circ N_F(X, Y) = 0; \quad X, Y \in \mathsf{X}(M).$$

In view of $F \circ N_F(X, Y) + N_F(FX, Y) = 0$, that can be immediately checked we can rewrite (22) also in the form

$$N_F(\pi^+ X, Y) = 0; \qquad X, Y \in \mathsf{X}(M). \tag{23}$$

On the other hand, by equality $[X, Y] = \nabla_X Y - \nabla_Y X - S(X, Y)$ and (16) it is easy to compute that

$$\begin{aligned} N_F(X, Y) &= -\alpha S(X, Y) - S(FX, FY) + F \circ S(FX, Y) + F \circ S(X, FY) \\ &= -F(F \circ S(X, Y) - S(FX, Y)) + (F \circ S(X, FY) - S(FX, FY)). \end{aligned} \tag{24}$$

We introduce tensor $U(X, Y) = F \circ S(X, Y) - S(FX, Y)$. Evidently, $F \circ U(X, Y) + U(FX, Y) = 0; \quad X, Y \in \mathsf{X}(M)$. Using the tensor we can rewrite (24) as follow $N_F(X, Y) = -F \circ U(X, Y) + U(X, FY)$. Then (23) will be written in the form $-F \circ U(\pi^+ X, Y) + U(\pi^+ X, FY) = 0$, and thus, $U(\pi^+ X, FY) = F \circ U(\pi^+ X, Y) = -U(F \circ \pi^+ X, Y) = -U(\lambda \pi^+ X, Y) = U(\pi^+ X, -\lambda Y)$, i.e. $U(\pi^+ X, (F + \lambda\,\mathrm{id})Y) = 0$. Note that $F^4 = \mathrm{id}$. Then we can rewrite the equality in the form $U(\pi^+ X, (\,\mathrm{id} + \lambda F^3)Y) = 0$, that is $U(\pi^+ X, \pi^+ Y) = 0$. Thus, distribution D_F^λ is involutive iff $U(\pi^+ X, \pi^+ Y) = 0; \quad X, Y \in \mathsf{X}(M)$. But it means that $F \circ S(\pi^+ X, \pi^+ Y) - S(F \circ \pi^+ X, \pi^+ Y) = 0$, i.e. $F \circ S(\pi^+ X, \pi^+ Y) =$

$\lambda S(\pi^+X, \pi^+Y)$, and therefore, $S(\pi^+X, \pi^+Y) \in D_F^\lambda$, i.e. $D_F^\lambda \subset \mathsf{V}$ is a subalgebra. Inversely, if $D_F^\lambda \subset \mathsf{V}$ is a subalgebra, then $F \circ S(\pi^+X, \pi^+Y) = \lambda S(\pi^+X, \pi^+Y) = S(\lambda\pi^+X, \pi^+Y) = S(F \circ \pi^+X, \pi^+Y)$, i.e. $U(\pi^+X, \pi^+Y) = 0; \quad X, Y \in \mathsf{X}(M)$ that is equivalent to distribution D_F^λ being involutive. If, in particular, D_F^λ is the principal fundamental distribution, then $\lambda S(\pi^+X, \pi^-Y) = S(\lambda\pi^+X, \pi^-Y) = S(F \circ \pi^+X, \pi^-Y) = S(\pi^+X, F \circ \pi^-Y) = S(\pi^+X, -\lambda\pi^-Y) = -\lambda S(\pi^+X, \pi^-Y)$, i.e. $S(\pi^+X, \pi^-Y) = 0; \quad X, Y \in \mathsf{X}(M)$ and then $S(\pi^+X, Y) = S(\pi^+X, \pi^+Y) \in D_F^\lambda$, i.e. $D_F^\lambda \subset \mathsf{V}$ is an ideal. $\square$

Definition 13. A πAQ_α-structure with integrable structural endomorphism I is called *semiholonomic.*

Evidently, the πAQ_α-structure being semiholonomic is equivalent to both its principal fundamental distributions being involutive.

Theorem 12. *πAQ_α-structure $\{\, I, J \,\}$ is semiholonomic iff*

$$I(X * Y) = I(X) * Y = X * I(Y); \qquad X, Y \in \mathsf{V}.$$

Proof. Note that integrability of endomorphism I is equivalent to it Nijenhuis tensor $N_I(X, Y)$ vanishing, that is, the following equality is valid: $\alpha S(X, Y) + S(IX, IY) - I \circ S(IX, Y) - I \circ S(X, IY) = 0; \quad X, Y \in \mathsf{X}(M)$. The equality can be rewritten by (16_3) in the form $\alpha S(X, Y) = I \circ S(IX, Y)$, or $I \circ S(X, Y) = S(IX, Y) = S(X, IY); \quad X, Y \in \mathsf{X}(M)$. Inversely, validity of these equalities implies the equality $N_I(X, Y) = 0; \quad X, Y \in \mathsf{X}(M)$, that is, integrability of endomorphism I. $\square$

Theorem 13. *A non-principal fundamental distribution D_F^λ of semiholonomic πAQ_α-structure is involutive iff $FX * FY = \lambda F(X * Y); \quad X, Y \in \mathsf{V}$. In particular, any of the structural endomorphisms J or K of semiholonomic πAQ_α-structure is integrable iff the adjoint algebra V is Abelian.*

Proof. Since the πAQ_α-structure is semiholomorphic, both its principal fundamental distributions are involutive. Let D_F^λ be involutive non-principal fundamental distribution. Then $D_F^\lambda \subset \mathsf{V}$ is a subalgebra. Note that $\mathsf{V} = D_I^\mu \oplus D_I^{-\mu}$ and that $D_I^\mu * D_I^{-\mu} = \{0\}$. Indeed, if $X \in D_I^\mu, Y \in D_I^{-\mu}$, then $\mu S(X, Y) = S(\mu X, Y) = S(IX, Y) = S(X, IY) = S(X, -\mu Y) = -\mu S(X, Y)$, i.e. $S(X, Y) = 0$. Now let $X, Y \in D_I^\mu$. Then $S(X + \lambda F^3 X, Y + \lambda F^3 Y) = S(X, Y) + \lambda S(F^3 X, Y) + \lambda S(X, F^3 Y) + \lambda^2 S(FX, FY)$. We show that $F(D_I^\mu) \subset D_I^{-\mu}$. Indeed, if $X \in D_I^\mu$, then $\mu F(X) = F(\mu X) = F \circ I(X) = -I \circ F(X)$, i.e. $F(X) = -\mu F(X)$ and thus, $F(X) \subset D_I^{-\mu}$. Consequently, $S(X + \lambda F^3 X, Y + \lambda F^3 Y) = S(X, Y) + \lambda^2 S(FX, FY)$. But $X + \lambda F^3 X = 2\pi^+ X \in D_F^\lambda$, and since $D_F^\lambda \subset \mathsf{V}$ is subalgebra, $S(X, Y) + \lambda^2 S(FX, FY) \in D_F^\lambda$. But since D_I^μ and $D_I^{-\mu}$ are also subalgebras, $S(X, Y) \in D_I^\mu$, $S(FX, FY) \in D_I^{-\mu}$, and hence,

$F \circ S(X,Y) \in D_I^{-\mu}$, $F \circ S(FX,FY) \in D_I^{\mu}$. In view of $\mathbf{K}_\alpha \otimes \mathsf{X}(M) = D_I^{\mu} \oplus D_I^{-\mu}$ we have: $F \circ S(X,Y) = \lambda^3 S(FX,FY)$, or $S(FX,FY) = \lambda F \circ S(X,Y)$. The same formula is valid for $X, Y \in D_I^{-\mu}$, and since $S(FX,FY) = 0 = \lambda F \circ S(X,Y)$; $\quad X \in D_I^{\mu}$, $\quad Y \in D_I^{-\mu}$, we have $S(FX,FY) = \lambda F(X,Y)$; $\quad X, Y \in \mathsf{X}(M)$, i.e. $FX * FY = \lambda F(X * Y)$; $\quad X, Y \in \mathsf{V}$. Inversely, if this equality is hold, then $F(X * Y) = \lambda^3 (FX) * (FY) = \lambda X * Y$; $\quad X, Y \in D_F^{\lambda}$, i.e. $D_F^{\lambda} \subset \mathsf{V}$ is a subalgebra, and D_F^{λ} is involutive distribution. If F is an integrable endomorphism, then both distributions, D_F^{λ} and $D_F^{-\lambda}$, are involutive, and thus, $F(X * Y) = \lambda^3 FX * FY$ and $F(X * Y) = -\lambda^3 FX * FY$ simultaneously, hence, $X * Y = 0$; $\quad X, Y \in \mathsf{V}$, i.e. V is Abelian algebra. Inversely, if V is Abelian algebra, then all fundamental distributions are involutive (since they are automatically subalgebras), in particular, distributions D_F^{λ} and $D_F^{-\lambda}$ are involutive, hence, endomorphism F is integrable. $\quad \square$

Corollary 1. *Giving three-web on a manifold is equivalent to giving on it a πAQ_α-structure $\{\, I, J \,\}$, its adjoint algebra V being J-linear, and endomorphism I being its involutory automorphism.*

Proof. Let a three-web, i.e. triples of n-dimensional involutive pair-wise complementary distributions, be given on $2n$-dimensional manifold M. The adjoint almost antiquaternionic structure of the three-web [35] is semiholonomic with respect to endomorphism J, distribution D_I^1 being involutive. By Theorems 12 and 13 its adjoint algebra is J-linear, and I its involutory automorphism. Inversely, if πAQ_α-structure $\{\, I, J \,\}$ has these properties, then, by automorphism I being involutory, the πAQ_α-structure is antiquaternionic, and, by the same theorems, distributions D_J^1, D_J^{-1} and D_I^1 define on M the structure of three-web. $\quad \square$

Corollary 2. *Integrability of any pair of structural endomorphisms of πAQ_α-structure $\{\, I, J \,\}$ implies integrability of the third structural endomorphism and is equivalent to the adjoint algebra being Abelian.*

Proof. It follows from the condition that at least one of the endomorphisms, I or J, is integrable. Redenoting, if necessary, I and J, we get that endomorphism I is integrable, i.e. πAQ_α-structure is semiholonomic. Since one of the endomorphisms, J or K, is also integrable, by Theorem 13 we have that the adjoint algebra is Abelian, and thus (by the same Theorem) all structural endomorphisms are integrable. $\quad \square$

Remark. The three-web whose Chern connection is torsion-free and thus the adjoint algebra is Abelian, is called *paratactic,* or *isoclinic-geodesic* [24].

3.3 Isoclinic distributions

Let $\{\, M, I, J \,\}$ be a πAQ_α-manifold, Q be its structural bundle.

Definition 14. An α-quaternion $q \in \{\, \mathsf{Q} \,\}$ whose kernel forms an r-dimensional distribution ($r \geq 0$) is called *admissible.*

Note that if $r > 0$, then q is an isotropic α-quaternion, i.e. $|q| = 0$. Indeed, in this case $\forall p \in M \, \exists X \in T_p(M) \mid X \neq 0 \ \&\ q(X) = 0$. But then $|q|^2 X = \overline{q} q(X) = 0$, and thus, $|q| = 0$ at arbitrary point $p \in M$, i.e. $|q| = 0$.

The examples of admissible α-quaternions with non-zero kernel are projectors π_F^+ and π_F^- with respect to structural endomorphism F; their kernels are fundamental distributions $D_F^{-\lambda}$ and D_F^{λ}, respectively. Now we set up the task of studying the kernel of arbitrary admissible α-quaternions. Further on all α-quaternions being regarded are assumed admissible.

Definition 15. The kernel of an isotropic α-quaternion is called *an isoclinic distribution* on πAQ_α-manifold.

Let q be an isotropic α-quaternion, $q = a\,\mathrm{id} + bI + cJ + dK$, $D = \ker q$ is the corresponding isoclinic distribution, $X \in D$. Note that $\mathbf{K}_\alpha \otimes \mathsf{X}(M) = D_I^{\lambda} \oplus D_I^{-\lambda}$, and thus, $X = X^+ + X^-$, $\quad X^+ \in D_I^{\lambda}$, $X^- \in D_I^{-\lambda}$. Then $q(X) = (a\,\mathrm{id} + bI + cJ + dK)(X^+ + X^-) = (a + b\lambda)X^+ + (c + d\lambda)JX^- + (c - d\lambda)JX^+ + (a - b\lambda)X^- = 0$, hence, $(a + b\lambda)X^+ + (c + d\lambda)JX^- = 0$, $(a - b\lambda)X^- + (c - d\lambda)JX^+ = 0$, or

$$\begin{cases} (a + b\lambda)X^+ + (c + d\lambda)JX^- = 0; \\ \alpha(c - d\lambda)X^+ + (a - b\lambda)JX^- = 0. \end{cases}$$

The determinant of the given system of equations is equal to $|q|^2 = 0$, and thus, vectors X^+ and JX^- are collinear. Let $X^+ \neq 0$. Then $JX^- = \nu X^+$, where $\nu = -\frac{a+b\lambda}{c+d\lambda}$ or $\nu = -\alpha\frac{c-d\lambda}{a-b\lambda}$. Similarly, if $X^- \neq 0$, i.e. $X \notin D_I^{\lambda}$, then $JX^+ = \nu X^-$, where $\nu = -\alpha\frac{c+d\lambda}{a+d\lambda}$ or $\nu = -\frac{a-b\lambda}{c-d\lambda}$. Thus,

$$D = \{\, X + \mu JX \mid X \in D_I^{\lambda} \,\},$$

where $\mu \in C^\infty(M)$ is fixed function, D_I^{λ} is one of the principal fundamental distributions. Distributions of such kind are called *isoclinic* in three-web theory [37].

Inversely, consider distribution of the type $D_\mu = \{\, X + \mu JX \mid \mu \in C^\infty(M),\ X \in D_I^{\lambda} \,\}$ on a πAQ_α-manifold M, and show, that it is isoclinic. If $\mu \neq \mathrm{const}$, we call it *a nontrivial isoclinic distribution.* We find all α-quaternions q, such that $D_\mu = \ker q$. Let $q = a\,\mathrm{id} + bI + cJ + dK$. Then

$$q(X + \mu IX) = 0 \iff \begin{cases} \alpha\mu(c + d\lambda) = -(a + b\lambda), \\ \mu(a - b\lambda) = -(c - d\lambda), \end{cases}$$

hence, $a = -\frac{1}{2}(\frac{1}{\mu} + \alpha\mu)c + \frac{1}{2}(\frac{1}{\mu} - \alpha\mu)d$, $b = \frac{1}{2\lambda}(\frac{1}{\mu} - \alpha\mu)c - \frac{1}{2}(\frac{1}{\mu} + \alpha\mu)d$, and thus,

$$q = \{-\frac{1}{2}(\frac{1}{\mu} + \alpha\mu)c + \frac{1}{2}(\frac{1}{\mu} - \alpha\mu)d\,\}\,\mathrm{id}\ +$$

$$
\begin{aligned}
&+\{\,\frac{1}{2\lambda}(\frac{1}{\mu}-\alpha\mu)c-\frac{1}{2}(\frac{1}{\mu}+\alpha\mu)d\,\}\,I+cJ+dK\\
=\ &\{-\frac{1}{2}(\frac{1}{\mu}+\alpha\mu)\,\mathrm{id}+\frac{1}{2\lambda}(\frac{1}{\mu}-\alpha\mu)I+J\,\}\,c\,+\\
&+\{\,\frac{1}{2}(\frac{1}{\mu}-\alpha\mu)\,\mathrm{id}-\frac{1}{2}(\frac{1}{\mu}+\alpha\mu)I+K\,\}\,d\,.
\end{aligned}
$$

Thus, distribution D_μ defined as a common kernel of α-quaternions forming a 2-dimensional submodule of structural bundle section module. Thus, we have proved

Theorem 14. *There exists a natural one-to-one correspondence between isoclinic distributions of a πAQ_α-structure and 2-dimensional distributions of isotropic α-quaternions of the mentioned type on a πAQ_α-manifold.* $\square$

It follows from the above that a unification of isoclinic distributions on a πAQ_α-manifold gives the foliation of cones over the manifold modelled by isotropic cones of corresponding α-quaternions. The elements of this foliation will be called *Segre cones* (as in three-web theory), and its points are called *isoclinic vectors*. Isoclinic subspaces at a given point of a πAQ_α-manifold are $\frac{1}{2}\dim M$-dimensional rulings of Segre cones at the point.

Definition 16. A πAQ_α-manifold will be called *isoclinic* if it admits an involutive isoclinic distribution different from fundamental distributions.

Let $\{\,M,I,J\,\}$ be a semiholonomic isoclinic πAQ_α-manifold, $q\in\mathsf{Q}$ be the isotropic α-quaternion defining involutive isoclinic distribution D_μ. Since D_μ is involutive, $\forall X,Y\in D_\mu\Longrightarrow Z=[X,Y]\in D_\mu$. By the proved above, $X=a+\mu Ja$, $Y=b+\mu Jb$, $Z=c+\mu Jc$, where $a,b,c\in D_I^\lambda$. Thus, $[X,Y]=[a+\mu Ja,b+\mu Jb]=[a,b]+\mu([a,Jb]+[Ja,b])+\mu^2[Ja,Jb]+a(\mu)Jb+\mu Ja(\mu)Jb-b(\mu)Ja-\mu Jb(\mu)Ja=c+\mu Jc$. Note that $N_J(a,b)=\alpha[a,b]+[Ja,Jb]-J[Ja,b]-J[a,Jb]$ and then $[a,Jb]+[Ja,b]=\alpha J(-N_J(a,b)+\alpha[a,b]+[Ja,Jb])=\alpha J(\alpha S(a,b)+S(Ja,Jb)+\alpha[a,b]+[Ja,Jb])=J\circ S(a,b)+\alpha J\circ S(Ja,Jb)+J[a,b]+\alpha J[Ja,Jb]$. Thus, $[a,b]+\mu J\circ S(a,b)+\alpha\mu J\circ S(Ja,Jb)+\mu J[a,b]+\alpha\mu J[Ja,Jb]+\mu^2[Ja,Jb]+a(\mu)Jb+\mu Ja(\mu)Jb-b(\mu)Ja-\mu Jb(\mu)Ja=c+Jc$. In view of $\mathbf{K}_\alpha\otimes\mathsf{X}(M)=D_I^\lambda\oplus D_I^{-\lambda}$, and distributions D_I^λ and $D_I^{-\lambda}$ being involutive and, thus, being subalgebras of the adjoint algebra, we get:

$$
\begin{aligned}
&1)\ [a,b]+\alpha\mu JS(Ja,Jb)+\alpha\mu J[Ja,Jb]=c,\\
&2)\ \mu JS(a,b)+\mu J[a,b]+\mu^2[Ja,Jb]+a(\mu)Jb+\\
&\quad+\mu Ja(\mu)Jb-b(\mu)Ja-\mu Jb(\mu)Ja=\mu Jc,
\end{aligned}
$$

and thus,

1) $c=[a,b]+\alpha\mu JS(Ja,Jb)+\alpha\mu J[Ja,Jb]$;

2) $c=[a,b]+S(a,b)+\alpha\mu J[Ja,Jb]+\dfrac{1}{\mu}a(\mu)b+Ja(\mu)b-\dfrac{1}{\mu}b(\mu)a-Jb(\mu)a.$

Substract elementwise one equality from the other: $S(a,b) - \alpha\mu J \circ S(Ja, Jb) = (\frac{1}{\mu}b(\mu) + Jb(\mu))a - (\frac{1}{\mu}a(\mu) + Ja(\mu))b$, or $J \circ S(a,b) - \mu S(Ja, Jb) = \omega(b)Ja - \omega(a)Jb$, where $\omega(c) = \frac{1}{\mu}(c+\mu Jc)(\mu) = \frac{1}{\mu}d\mu\,(c+\mu Jc) = d\,(\ln|\mu|)\circ \mathrm{id}+\mu J)c$. Evidently, the inverse is also true: this condition implies that isoclinic distribution D_μ is involutive. Thus, we have proved

Theorem 15. *A semiholonomic πAQ_α-manifold $\{\,M, I, J\,\}$ is isoclinic iff*

$$\exists \mu \in C^\infty(M); \mu \neq \pm 1 \;\&\; J(X*Y) - \mu(JX*JY) = \omega(Y)JX - \omega(X)JY;$$

where $X, Y \in D_I^\lambda,\ \omega = d\,(\ln|\mu|)(\,\mathrm{id} + \mu J)$. In this case $D = \{\,X + \mu JX \mid X \in D_I^\lambda\,\}$ is an involutive isoclinic distribution on M. $\square$

Corollary 1. *A semiholonomic πAQ_α-manifold $\{\,M, I, J\,\}$ having not less than three involutive fundamental distributions is isoclinic iff*

$$X*Y = \frac{1}{1-\mu}\{\,\omega(Y)X - \omega(X)Y\,\}; \qquad X, Y \in D_I^\lambda. \quad \square$$

The above result generalizes the known criterion of M.A.Akivis of three-web being isoclinic [37].

Corollary 2. *A semiholonomic πAQ_α-manifold having more than three involutive fundamental distributions is isoclinic.* $\square$

Definition 17. A πAQ_α-manifold is called *isoclinic-geodesic* if it admits a non-trivial totally geodesic (in canonical connection) isoclinic distribution.

Let $\{\,M, I, J\,\}$ be a πAQ_α-manifold, $q \in \{\,\mathsf{Q}\}$ be an isotropic α-quaternion defining isoclinic distribution $D = \{\,X + \mu JX \mid X \in D_I^\lambda\,\}$. Let $X, Y \in D,\ X = a + \mu Ja,\ Y = b + \mu Jb,\ a, b \in D_I^\lambda$. Then $\nabla_X Y = \nabla_a b + \mu\nabla_{Ja} b + \mu\nabla_a (Jb) + \mu^2\nabla_{Ja}(Jb) + a(\mu)Jb + \mu Ja(\mu)Jb = \nabla_a b + \mu J\nabla_a b + \mu(\nabla_{Ja} b + \mu J\nabla_{Ja} b) + a(\mu)Jb + \mu Ja(\mu)Jb$. It is clear that D is totally geodesic, i.e. $\nabla_X Y \in D \quad (X, Y \in D)$ iff $a(\mu) + \mu Ja(\mu) = 0$, i.e., in the notations of the above theorem, $\omega = 0$. We have proved

Theorem 16. *A πAQ_α-manifold is isoclinic-geodesic iff $\omega = 0$ where*

$$\omega = d\,(\ln|\mu|)\circ(\,\mathrm{id} + \mu J); \quad \mu \in C^\infty(M);\ \mu \neq const.$$

In this case $D = \{\,X + \mu JX \mid X \in D_I^\lambda\,\}$ is a totally geodesic isoclinic distribution on this manifold. $\square$

Now let $\{\,M, I, J\,\}$ be a semiholonomic πAQ_α-manifold, D be an involutive isoclinic distribution on $M,\ X, Y \in D,\ X = a + \mu Ja,\ Y = b + \mu Jb;\ a, b \in D_I^\lambda$. Then $X*Y = a*b + \mu(Ja*b + a*Jb) + \mu^2(Ja*Jb)$. In view of Theorem 11, D_I^λ and D_I^{-l} are ideals of the adjoint algebra, and since $J(D_I^\lambda) \subset D_I^{-\lambda}$, $Ja*b = a*Jb = 0$.

Thus, $X * Y \in D \iff a * b + \mu^2(Ja * Jb) = c + \mu Jc$; $(c \in D_I^\lambda)$, hence, $c = a * b, Jc = \mu(Ja * Jb)$, and consequently, $J(a * b) = \mu(Ja * Jb)$; $(a, b \in D_I^\lambda)$. Evidently, the inverse is also true, i.e. there holds

Theorem 17. *An involutive isoclinic distribution $D = \{ X + \mu JX \mid X \in D_I^\lambda \}$ of a semiholonomic πAQ_α-manifold M is a subalgebra of the adjoint algebra iff*

$$J(X * Y) = \mu(JX * JY); \qquad X, Y \in D_I^\lambda. \quad \square$$

Further, by Theorems 15 and 16 we get

Theorem 18. *A semiholonomic isoclinic πAQ_α-manifold M is isoclinic-geodesic iff $J(X * Y) = \mu(JX * JY)$; $\quad X, Y \in D_I^\lambda$; $\mu \in C^\infty(M); \mu \neq const.$ In this case $D = \{ X + \mu JX \mid X \in D_I^\lambda \}$ is a totally geodesic isoclinic distribution on M being a subalgebra of the adjoint algebra.* $\square$

Corollary. *A semiholonomic isoclinic πAQ_α-manifold having not less than three involutive fundamental distributions is isoclinic-geodesic iff its adjoint algebra is Abelian.*

Proof. It immediately follows from Theorem 16 and Corollary 1 of Theorem 15. $\square$

The result shows that the notion of an isoclinic-geodesic πAQ_α-structure is the generalization of the notion of an isoclinic-geodesic three-web, mentioned above and, moreover, it explains the geometric sense of the notion.

3.4 Hermitian and Einsteinian metrics generated by antiquaternionic structure on pseudo-Riemannian manifold

Let $(M, (\cdot,\cdot))$ be a pseudo-Riemannian manifold, $\{ I, J \}$ be a πAQ_α-structure on M. We define the pseudo-Riemannian metric g on M by the formula

$$g(X, Y) = (X, Y) + (IX, IY) + (JX, JY) + (KX, KY); \qquad X, Y \in \mathsf{X}(M).$$

Evidently,

$$\begin{aligned} g(IX, IY) &= g(X, Y); \qquad & g(IX, Y) &= \alpha g(X, IY); \\ g(JX, JY) &= g(X, Y); \qquad & g(JX, Y) &= \alpha g(X, JY); \\ g(KX, KY) &= g(X, Y); \qquad & g(KX, Y) &= -g(X, KY). \end{aligned} \tag{25}$$

Let $\alpha = +1$, i.e. the πAQ_α-structure under consideration is almost antiquaternionic. Then two neutral metrics $g_1(X, Y) = g(X, IY)$ and $g_2(X, Y) = g(X, JY)$

are defined on M in addition. Indeed,

$$g_1(JX,JY)=g(JX,IJY)=-g(JX,JXY)=-g(X,IY)=-g_1(X,Y);$$
$$g_2(IX,IY)=g(IX,JIY)=-g(IX,IJY)=-g(X,JY)=-g_2(X,Y).$$

Thus, whole metrics sheaf

$$\langle X,Y\rangle = g(X,Y)+\lambda g(X,IY)+\mu g(X,JY) \tag{26}$$

is associated with g. It is easy to see that the metrics are non-degenerate iff $\lambda^2+\mu^2\neq 1$. Indeed, let $\langle X,Y\rangle=0;\quad X,Y\in \mathsf{X}(M)$. Then by non-degeneracy of g we have:

$$X+\lambda IX+\mu JX=0. \tag{27}$$

Acting on both parts of the equality at first by endomorphism I and then by J we have respectively:

$$\lambda X+IX+\mu KX=0;\qquad \mu X+JX-\lambda KX=0.$$

We multiply both parts of the first equality on λ, and of the second one by μ and add the received equalities term by term:

$$(\lambda^2+\mu^2)X+\lambda IX+\mu JX=0.$$

In view of (27) we see that either $\lambda^2+\mu^2=1$, or $X=0$. We find all purely imaginary antiquaternions J forming almost Hermitian structure with a metric of such kind. Let $J=\beta I+\gamma J+\delta K$. By direct calculations we find that $\{\,J,\langle\cdot,\cdot\rangle\,\}$ is almost Hermitian structure on M, i.e. $\langle JX,JY\rangle=\langle X,Y\rangle;\quad X,Y\in\mathsf{X}(M)$ iff the following equation system is fulfilled:

$$\begin{cases}\beta^2+\gamma^2+\delta^2+2\mu\beta\gamma=1;\\ -2\gamma\delta+2\mu\beta\gamma+\lambda(\beta^2-\gamma^2-\delta^2)=\lambda;\\ 2\beta\delta+\mu(-\beta^2+\gamma^2-\delta^2)+2\lambda\beta\gamma=\mu;\\ -\beta^2-\gamma^2+\delta^2=1.\end{cases}$$

According to the last system equation, $\beta^2+\gamma^2=\delta^2-1$. Therefore, the system can be rewritten in the form:

$$\begin{cases}\delta\varphi=1;\\ \gamma\varphi=-\lambda;\\ \beta\varphi=\mu;\\ \beta^2+\gamma^2=\delta^2-1,\end{cases}$$

where $\varphi=\delta-\mu\beta+\lambda\gamma$. We have from the first triple equations:

$$\varphi^2(-\beta^2-\gamma^2+\delta^2)=1-\lambda^2-\mu^2.$$

Note that in view of the first system equation $\delta \neq 0$ and hence, $\frac{\gamma}{\delta} = -\lambda$, $\frac{\beta}{\delta} = \mu$, i.e. $\beta = \mu\delta$, $\gamma = -\lambda\delta$, and in view of the last system equation,

$$\beta = \pm\frac{\mu}{\sqrt{1-\lambda^2-\mu^2}}; \quad \gamma = \pm\frac{\lambda}{\sqrt{1-\lambda^2-\mu^2}}; \quad \delta = \pm\frac{1}{\sqrt{1-\lambda^2-\mu^2}}.$$

Thus we have proved

Theorem 19. *Every pseudo-Riemannian metric $(\cdot,\cdot)$ on πAQ_1-manifold M is associated with two-parameter family of almost Hermitian structures $\{\mathcal{J}, \langle\cdot,\cdot\rangle\}$ on M ;*

$$\mathcal{J} = \pm\frac{\mu I - \lambda J + K}{\sqrt{1-\lambda^2-\mu^2}}; \qquad \langle X,Y\rangle = g(X,Y) + \lambda g(X,IY) + \mu g(X,JY);$$

$$\lambda, \mu \in \mathbf{R}, \quad \lambda^2 + \mu^2 < 1;$$

where $g(X,Y) = (X,Y) + (IX,IY) + (JX,JY) + (KX,KY);\ X,Y \in \mathsf{X}(M)$. □

Remark. In the case $\lambda^2 + \mu^2 > 1$ every pseudo-Riemannian metric is associated with two-parameter family of hyperbolic almost Hermitian structures $\{\mathcal{J}, \langle\cdot,\cdot\rangle\}$, where

$$\mathcal{J} = \pm\frac{\mu I - \lambda J + K}{\sqrt{\lambda^2+\mu^2-1}}, \qquad \langle X,Y\rangle = g(X,Y) + \lambda g(X,IY) + \mu g(X,JY).$$

Indeed, in the case $\mathcal{J}^2 = \mathrm{id}$, i.e. $\|\mathcal{J}\|^2 = -1$, and therefore, $-\beta^2 - \gamma^2 + \delta^2 = -1$, hence, $\delta = \pm\frac{1}{\sqrt{\lambda^2+\mu^2-1}}$.

Now let M be parallelizable locally homogeneous naturally reductive manifold, $(\cdot,\cdot)$ be invariant pseudo-Riemannian metric on M, G be fundamental group of local isometries of M, $H \subset G$ be isotropy subgroup, g and h be Lie algebras of Lie groups G and H, respectively. Then $\mathsf{g} = \mathsf{h} \oplus \mathsf{m}$, $\mathrm{ad}(\mathsf{h})\mathsf{m} \subset \mathsf{m}$. As usual, we shall identify a tangent space $T_p(M)$; $p \in M$, and subspace $\mathsf{m} \subset \mathsf{g}$ in canonical way. Natural reductivity means that $([X,Y]_{\mathsf{m}}, Z) = (X, [Y,Z]_{\mathsf{m}})$; $X,Y,Z \in \mathsf{m}$. Recall [35] that manifold $M \times M$ carries naturally defined πAQ_1-structure $\{I, J\}$. The distributions D_I^1 and D_I^{-1} of the structure are generated by subspaces m_1 and m_2 respectively, where m_i is the i-th term of direct sum $\mathsf{m} \oplus \mathsf{m}$ corresponding to decomposition $T_{(p,q)}(M \times M) = T_p(M) \oplus T_q(M)$. Since distributions D_I^1 and D_I^{-1} are involutive the πAQ_1-structure is semiholonomic. Moreover, let us denote the diagonal of manifold $M \times M$ by Δ. Then the distribution D_J^1 is generated by subspace m_0 corresponding to $T_p(\Delta)$; $p \in \Delta$, hence, it is involutive also. Therefore, for the stated πAQ_1-structure we have:

$$I(X) * Y = X * I(Y) = I(X * Y); \qquad J(X * Y) = J(X) * J(Y); \tag{28}$$

where $X, Y \in \mathsf{X}(M \times M)$. Let us calculate a torsion tensor S of the canonical connection $\tilde{\nabla}$ for the πAQ_1-structure. Note that $[X, Y] = 0; \quad X \in \mathsf{m}_1, Y \in \mathsf{m}_2$, and in view of (19) we have: $\tilde{\nabla}_X Y = 0; \quad X, Y \in \mathsf{m} \oplus \mathsf{m}$. Therefore, $X * Y = S(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = -[X, Y]; \quad X, Y \in \mathsf{m} \oplus \mathsf{m}$. Here and later on by $[X, Y]$ we mean of vectors X and Y commutator restriction on $\mathsf{m} \oplus \mathsf{m} \subset \mathsf{g} \oplus \mathsf{g}$. Consequently, relations (28) take the form:

$$[I(X), Y] = [X, I(Y)] = I([X, Y]); \qquad J([X, Y]) = [J(X), J(Y)]; \tag{29}$$

where $X, Y \in \mathsf{m} \oplus \mathsf{m}$.

We calculate Riemannian connection ∇ of the metric $\overline{g} = \langle \cdot, \cdot \rangle$ generated by direct product metric of manifold $M \times M$ as mentioned previously. We have:

$$2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle - \langle [Y, Z], X \rangle.$$

After direct calculations with respect to natural reductivity of manifold we get:

$$Z + \lambda IZ + \mu JZ = A, \tag{30}$$

where $Z = 2\nabla_X Y$, $A = [X, Y] + \mu J[X, Y] + \mu[X, JY] - \mu[JX, Y] + \lambda I[X, Y]$. Acting on (30) by endomorphisms I, J and K we receive correspondingly:

$$\begin{aligned} &1)\ \lambda Z + IZ + \mu KZ = IA; \\ &2)\ \mu Z + JZ - \lambda KZ = JA; \\ &3)\ KZ - \lambda JZ + \mu IZ = KA. \end{aligned} \tag{31}$$

Multiply the third equation in (31) first by μ and then by λ:

$$\begin{aligned} &1)\ \mu KZ - \mu\lambda JZ + \mu^2 IZ = \mu KA; \\ &2)\ \lambda KZ - \lambda^2 JZ + \lambda\mu IZ = \lambda KA. \end{aligned} \tag{32}$$

Subtract (32_1) from (31_1) termwise and add (32_2) and (31_2) termwise:

$$\begin{aligned} &1)\ \lambda Z + (1 - \mu^2) IZ + \lambda\mu JZ = (I - \mu K)A; \\ &2)\ \mu Z + \lambda\mu IZ + (1 - \lambda^2) JZ = (J + \lambda K)A. \end{aligned} \tag{33}$$

Now multiplying (30) first by λ and then by μ we receive, correspondingly:

$$\begin{aligned} &1)\ \lambda Z + \lambda^2 IZ + \lambda\mu JZ = \lambda A; \\ &2)\ \mu Z + \lambda\mu IZ + \mu^2 JZ = \mu A; \end{aligned} \tag{34}$$

Subtract (34_1) from (33_1) and (34_2) from (33_2) termwise:

$$Z = \frac{\mathrm{id} - \lambda I - \mu J}{1 - \lambda^2 - \mu^2} A,$$

therefore,

$$\nabla_X Y = \frac{1}{2}\frac{\mathrm{id} - \lambda I - \mu J}{1-\lambda^2-\mu^2}\{\,[X,Y] + \mu J[X,Y] + \mu[X,JY] - \mu[JX,Y] + \lambda[X,Y]\,\}.$$

Simplifying the equation with regard to (29) we get finally:

$$\begin{aligned}\nabla_X Y = {} & \frac{1}{2}[X,Y] + \frac{\mu^2+\mu}{2(1-\lambda^2-\mu^2)}([X,JY]-[JX,Y]) + \\ & + \frac{\lambda\mu}{2(1-\lambda^2-\mu^2)}([KX,Y]-[X,KY]).\end{aligned} \tag{35}$$

With respect to (35) we calculate covariant differential of tensors I, J, K and $\mathcal{J}$ in Riemannian connection: $\nabla_X (I)Y = \nabla_X (IY) - I(\nabla_X Y)$. After direct calculations with regard to (29) we get:

$$\nabla_X (I)Y = -\frac{\mu^2+\mu}{1-\lambda^2-\mu^2}[X,KY] + \frac{\lambda\mu}{1-\lambda^2-\mu^2}[X,JY].$$

Similarly,

$$\begin{aligned}\nabla_X (J)Y = {} & \frac{1}{2}\{\,[X,JY] - J[X,Y]\,\} + \\ & + \frac{\mu^2+\mu}{2(1-\lambda^2-\mu^2)}\{\,[X,Y] - J[X,Y] - [JX,Y] + [X,JY]\,\} + \\ & + \frac{\lambda\mu}{2(1-\lambda^2-\mu^2)}\{-I[X,Y] + K[X,Y] - [KX,Y] + [X,KY]\,\}; \\ \nabla_X (K)Y = {} & \frac{1}{2}\{\,[X,KY] - K[X,Y]\,\} + \\ & + \frac{\mu^2+\mu}{2(1-\lambda^2-\mu^2)}\{-I[X,Y] - K[X,Y] - [KX,Y] + [X,KY]\,\} + \\ & + \frac{\lambda\mu}{2(1-\lambda^2-\mu^2)}\{\,[X,Y] + J[X,Y] - [JX,Y] + [X,JY]\,\}.\end{aligned}$$

Taking into account the definition of tensor $\mathcal{J}$ we receive from here:

$$\begin{aligned}\nabla_X (\mathcal{J})Y = {} & \frac{1}{2(1-\lambda^2-\mu^2)}\{\,\lambda\mu^2[X,Y] + (\mu\lambda^2-\mu^2-\mu)I[X,Y] + \\ & + (\lambda-\lambda^3+2\lambda\mu)J[X,Y] + (\lambda^2-\lambda^2\mu-\mu-1)K[X,Y] + \\ & + (\lambda^3+2\lambda\mu^2-\lambda)[X,JY] + \lambda\mu^2[JX,Y] + (1+\mu-2\mu^3 - \\ & - 2\mu^2-\mu\lambda^2-\lambda^2)[X,KY] + (\lambda^2\mu-\mu^2-\mu)[KX,Y]\,\}.\end{aligned}$$

In particular, the structure $\{\,\mathcal{J}, \langle\cdot,\cdot\rangle\,\}$ is nearly Kaehlerian, i.e. $\nabla_X (\mathcal{J})X = 0;\quad X \in \mathsf{X}(M)$ iff

$$(\lambda^3+\lambda\mu^2-\lambda)[X,JX] + (1+2\mu-2\mu^3-\mu^2-2\lambda^2\mu-\lambda^2)[X,KX] = 0,$$

i.e.
$$\begin{cases} 1)\ \lambda(\lambda^2+\mu^2-1)=0; \\ 2)\ (1+2\mu)(\lambda^2+\mu^2-1)=0, \end{cases}$$
and since $\lambda^2+\mu^2\neq 1$, we have: $\lambda=0,\ \mu=-\frac{1}{2}$. We get the following result:

Theorem 20. *Almost Hermitian structure $\{\mathcal{J},\langle\cdot,\cdot\rangle\}$ on manifold $M\times M$, described above is nearly Kaehlerian iff $\lambda=0,\mu=-\frac{1}{2}$, i.e.*
$$\mathcal{J}=\pm\frac{I-2K}{\sqrt{3}};\qquad \langle X,Y\rangle=g(X,Y)-\frac{1}{2}g(X,JY).\quad\square$$

Now we calculate the compositional tensor of adjoint Q-algebra [28] of almost Hermitian structure $\{\mathcal{J},\langle\cdot,\cdot\rangle\}$ on the manifold $M\times M$ described above. After direct calculations we get: $\nabla_X(\mathcal{J})\mathcal{J}Y+\nabla_{\mathcal{J}X}(\mathcal{J})Y=(2b\mu-e\lambda-f\lambda-h-r)[X,Y]+(2a\mu-h\lambda-r\lambda-e-f)I[X,Y]-(2d\mu-e\lambda+f\lambda-h-r)J[X,Y]-(2c\mu+h\lambda+r\lambda-f-e)K[X,Y]-(c\lambda+a\lambda-b+d)[JX,Y]-(a\lambda+c\lambda-b+d)[X,JY]-(d\lambda+b\lambda-a+c)[KX,Y]-(b\lambda+d\lambda-a+c)[X,KY]$, where
$$a=-\frac{\lambda\mu^2}{2\sqrt{(1-\lambda^2-\mu^2)^3}};\ b=\frac{\lambda^2\mu-\mu^2-\mu}{2\sqrt{(1-\lambda^2-\mu^2)^3}};\ c=\frac{\lambda-\lambda^3+2\lambda\mu}{2\sqrt{(1-\lambda^2-\mu^2)^3}};$$
$$d=\frac{\lambda^2-\lambda^2\mu-\mu-1}{2\sqrt{(1-\lambda^2-\mu^2)^3}};\ e=\frac{\lambda^3+2\lambda\mu^2-\lambda}{2\sqrt{(1-\lambda^2-\mu^2)^3}};\ f=\frac{\lambda\mu^2}{2\sqrt{(1-\lambda^2-\mu^2)^3}};$$
$$h=\frac{1+\mu-2\mu^3-2\mu^2-\mu\lambda^2-\lambda^2}{2\sqrt{(1-\lambda^2-\mu^2)^3}};\ r=\frac{\mu\lambda^2-\mu^2-\mu}{2\sqrt{(1-\lambda^2-\mu^2)^3}}.$$
In particular, the tensor is skew-symmetric by covariant arguments, hence, almost Hermitian structure $\{\mathcal{J},\langle\cdot,\cdot\rangle\}$ belongs to class G_1 in Gray-Hervella classification [28]. Note also, that $2b\mu-e\lambda-f\lambda-h-r=(\lambda^2+\mu^2-1)(\lambda^2-1)$. Since $\lambda^2\leq\lambda^2+\mu^2<1$, the coefficient at $[X,Y]$ is nonzero, hence, almost Hermitian structure $\{\mathcal{J},\langle\cdot,\cdot\rangle\}$ is nonintegrable, if M differs from locally symmetric space. Besides we get that $\{\mathcal{J},\langle\cdot,\cdot\rangle\}$ is quasi-Kaehlerian structure, i.e. $\nabla_{\mathcal{J}X}(\mathcal{J})\mathcal{J}Y+\nabla_X(\mathcal{J})Y=0$, iff it is nearly Kaehlerian, i.e. $\lambda=0,\ \mu=-\frac{1}{2}$. Formulate the results obtained:

Theorem 21. *Let $\{M,(\cdot,\cdot)\}$ be a parallelizable locally homogeneous naturally reductive Riemannian manifold. Then the canonical πAQ_1-structure inherently generates two-parameter family of almost Hermitian structures on manifold $M\times M$. The structures belong to class G_1. If M differs from locally symmetric space the structures are nonintegrable. The family contains the only (up to sign) quasi-Kaehlerian structure $\{\mathcal{J}_0,\langle\cdot,\cdot\rangle_0\}$ at $\lambda=0,\ \mu=-\frac{1}{2}$, which is a nearly Kaehlerian structure.* $\square$

Theorem 21 gives a wide spectrum of examples of nonintegrable almost Hermitian G_1-structures, as well as examples of proper (i.e. different from Kaehlerian) nearly Kaehlerian structures. Note that up to now only a limited number of examples of the mentioned structures is known.

Returning to considering metrics $\langle X,Y\rangle = g(X,Y)+\lambda g(X,IY)+\mu g(X,JY)$ on manifold $M\times M$ we calculate Riemann-Christoffel tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

of the metrics. Direct though cumbersome calculations taking into account (35) give the following result:

$$\begin{aligned}
R(X,Y)Z = {} & \frac{1}{4}[[X,Y],Z] + \frac{\mu^2+\mu}{4(1-\lambda^2-\mu^2)}\{\,[X,[JY,JZ]] - [Y,[JX,JZ]]\,\} - \\
& -\frac{\mu^2+\mu}{4(1-\lambda^2-\mu^2)}\{\,[JX,[Y,Z]] - [JY,[X,Z]]\,\} + \\
& +\frac{\lambda\mu}{4(1-\lambda^2-\mu^2)}\{\,[KX,[Y,Z]] - [KY,[X,Z]]\,\} - \\
& -\frac{\lambda\mu}{4(1-\lambda^2-\mu^2)}\{\,[X,[KY,JZ]] - [Y,[KX,JZ]]\,\} + \\
& +\frac{\mu^2+\mu}{4(1-\lambda^2-\mu^2)}\{\,[X,[Y,JZ]] - [Y,[X,JZ]]\,\} + \\
& +\frac{(\mu^2+\mu)^2}{4(1-\lambda^2-\mu^2)^2}\{\,[X,[JY,Z]] - [Y,[JX,Z]]\,\} - \\
& -\frac{(\mu^2+\mu)^2}{4(1-\lambda^2-\mu^2)^2}\{\,[JX,[Y,JZ]] - [JY,[X,JZ]]\,\} + \\
& +\frac{(\mu^2+\mu)\mu\lambda}{4(1-\lambda^2-\mu^2)^2}\{\,[KX,[Y,JZ]] - [KY,[X,JZ]]\,\} - \\
& -\frac{\mu^2+\mu}{4(1-\lambda^2-\mu^2)}\{\,[X,[JY,Z]] - [Y,[JX,Z]]\,\} - \\
& -\frac{(\mu^2+\mu)^2}{4(1-\lambda^2-\mu^2)^2}\{\,[X,[Y,JZ]] - [Y,[X,JZ]]\,\} + \\
& +\frac{(\mu^2+\mu)^2}{4(1-\lambda^2-\mu^2)^2}\{\,[JX,[JY,Z]] - [JY,[JX,Z]]\,\} - \qquad (36) \\
& -\frac{(\mu^2+\mu)\mu\lambda}{4(1-\lambda^2-\mu^2)^2}\{\,[KX,[JY,Z]] - [KY,[JX,Z]]\,\} + \\
& +\frac{\lambda\mu}{4(1-\lambda^2-\mu^2)}\{\,[X,[KY,Z]] - [Y,[KX,Z]]\,\} + \\
& +\frac{(\lambda\mu)^2}{4(1-\lambda^2-\mu^2)^2}\{\,[KX,[KY,Z]] - [KY,[KX,Z]]\,\} - \\
& -\frac{\mu^2+\mu}{2(1-\lambda^2-\mu^2)}\{\,[[X,Y],JZ] - [[JX,JY],Z]\,\} + \\
& +\frac{\lambda\mu}{2(1-\lambda^2-\mu^2)}\{\,[[X,Y],KZ] - [[KX,JY],Z]\,\} -
\end{aligned}$$

$$
\begin{aligned}
&-\frac{(\mu^2+\mu)\mu\lambda}{4(1-\lambda^2-\mu^2)}\{\,[JX,[KY,Z]]-[JY,[KX,Z]]\,\}+\\
&+\frac{(\lambda\mu)^2}{4(1-\lambda^2-\mu^2)^2}\{\,[X,[Y,JZ]]-[Y,[X,JZ]]\,\}-\\
&-\frac{\lambda\mu}{4(1-\lambda^2-\mu^2)}\{\,[X,[Y,KZ]]-[Y,[X,KZ]]\,\}+\\
&+\frac{(\mu^2+\mu)\mu\lambda}{4(1-\lambda^2-\mu^2)}\{\,[JX,[Y,KZ]]-[JY,[X,KZ]]\,\}-\\
&-\frac{(\lambda\mu)^2}{4(1-\lambda^2-\mu^2)^2}\{\,[KX,[Y,KZ]]-[KY,[X,KZ]]\,\}-\\
&-\frac{(\lambda\mu)^2}{4(1-\lambda^2-\mu^2)^2}\{\,[X,[JY,Z]]-[Y,[JX,Z]]\,\}.
\end{aligned}
$$

With regard to the identity we shall calculate Ricci tensor $r(X)=g^{ij}\,R(X,e_i)e_j$ of the manifold. We choose a base $\{\,e_a,e_{\hat a}\,\}$ of a space $T_p(M\times M)$, $\quad p\in M$, where $e_a\in(D^1_I)_p$, $e_{\hat a}\in(D^{-1}_I)_p$, $e_{\hat a}=J(e_a)$, $\quad a=1,\dots,n=\dim M$, $\{\,e_1,\dots$ $\dots,e_n\,\}$ is orthonormalized base of subspace $\mathsf{m}\subset\mathsf{g}$ relative to metric $(\cdot,\cdot)$ under relevant identifications. Evidently, in the base

$$
\begin{aligned}
&(\overline{g}_{ij})=\begin{pmatrix}(1+\lambda)I_n & \mu I_n\\ \mu I_n & (1-\lambda)I_n\end{pmatrix};\\
&(\overline{g}^{ij})=\tfrac{1}{1-\lambda^2-\mu^2}\begin{pmatrix}(1-\lambda)I_n & \mu I_n\\ -\mu I_n & (1+\lambda)I_n\end{pmatrix};
\end{aligned}
$$

where I_n is the unit matrix of order n. Therefore, we have:

$$
\begin{aligned}
r(X)=\frac{1}{1-\lambda^2-\mu^2}\sum_{a=1}^{n}\{\,&(1-\lambda)R(X,e_a)e_a+(1+\lambda)R(X,e_{\hat a})e_{\hat a}-\\
&-\mu R(X,e_a)e_{\hat a}-\mu R(X,e_{\hat a})e_a\,\}.
\end{aligned}
$$

A direct calculation in view of (36) shows that

$$
\begin{aligned}
r(X)=\frac{1}{1-\lambda^2-\mu^2}\sum_{a=1}^{n}\{\,&(-\frac{1-\lambda}{4}+\frac{\mu^2(\mu-\lambda+1)}{4(1-\lambda^2-\mu^2)})[[X,e_a],e_a]+\\
&+(-\frac{1+\lambda}{4}+\frac{\mu^2(\lambda+\mu+1)}{4(1-\lambda^2-\mu^2)})[[X,e_{\hat a}],e_{\hat a}]+\\
&+\frac{\mu(-2\mu^2-\lambda^2+3\lambda\mu-3\mu+2\lambda-1)}{4(1-\lambda^2-\mu^2)}[[JX,e_a],e_a]-\\
&-\frac{\mu(\lambda^2+2\mu^2+3\lambda\mu+2\lambda+3\mu+1)}{4(1-\lambda^2-\mu^2)}[[JX,e_{\hat a}],e_{\hat a}].
\end{aligned}
\tag{37}
$$

Lemma 2. *Let G is semisimple Lie group. Then the equality is valid:*

$$g^{ij}[[X,e_i],e_j]=-X,$$

where $\{e_1,\dots,e_n\}$ is some base of its Lie algebra $\mathfrak{g}$, (g^{ij}) is a contravariant metric tensor for Killing metric $\mathcal{K}$.

Proof. We have by definition: $\mathcal{K}(X,Y)=-\operatorname{tr}(\operatorname{ad}X\circ\operatorname{ad}Y)$, $\quad X,Y\in\mathfrak{g}$. Evidently, if $L:V\to V$ is any endomorphism of Euclidean space $(V,\langle\cdot,\cdot\rangle)$, then $\operatorname{tr}L=g^{ij}\langle L(e_i),e_j\rangle$. Indeed, $g^{ij}\langle L(e_i),e_j\rangle=g^{ij}\,L^k{}_i\,\langle e_k,e_j\rangle=g^{ij}\,g_{kj}\,L^k{}_i=L^i{}_i$. Therefore,

$$\begin{aligned}\mathcal{K}(X,Y)&=-g^{ij}\langle[X,[Y,e_i]],e_j\rangle=g^{ij}\langle[Y,e_i],[X,e_j]\rangle=\\&=-g^{ij}\langle Y,[[X,e_j],e_i]\rangle=-\langle g^{ij}[[X,e_i],e_j],Y\rangle\end{aligned}$$

and hence $\mathcal{K}(X)=-g^{ij}[[X,e_i],e_j]$ is the result of subscript raising of Killing form by itself, i.e. $\mathcal{K}=\operatorname{id}$ and hence $g^{ij}[[X,e_i],e_j]=-X$. $\square$

Corollary. *Let $\{e_i\}$ be an orthonormalized (with respect to Killing form) base of semisimple Lie algebra. Then*

$$\sum_i[[X,e_i],e_i]=-X.\quad\square$$

We return to considering metric $\langle\cdot,\cdot\rangle$ on manifold $M\times M$, where $M=G$ is semisimple Lie group equipped with Killing metric. Introduce the notations

$$\begin{aligned}A&=-\frac{1-\lambda}{4}+\frac{\mu^2(\mu-\lambda+1)}{4(1-\lambda^2-\mu^2)};\\B&=-\frac{1+\lambda}{4}+\frac{\mu^2(\mu+\lambda+1)}{4(1-\lambda^2-\mu^2)};\\C&=\frac{\mu(-2\mu^2-\lambda^2+3\lambda\mu-3\mu+2\lambda-1)}{4(1-\lambda^2-\mu^2)};\\D&=-\frac{\mu(2\mu^2+\lambda^2+3\lambda\mu+3\mu+2\lambda+1)}{4(1-\lambda^2-\mu^2)}.\end{aligned}$$

Let $\langle\cdot,\cdot\rangle$ be Einsteinian metric, i.e. $r=\varepsilon\operatorname{id}$. According to (37) and Lemma 2 we have in this case: $A=B$, $C=D=0$ and hence $C-D=0$, i.e. $\lambda\mu(3\mu+2)=0$. We the have following possibilities:

1. $\mu=0$. Then $C=D=0$, $A=-\frac{1-\lambda}{4}$, $B=-\frac{1+\lambda}{4}$. Therefore $\lambda=0$, i.e. $\langle X,Y\rangle=\mathcal{K}(X,Y)$.
2. $\mu\neq0$, $\lambda=0$. Then $2\mu^2+3\mu+1=0$, i.e. $(2\mu+1)(\mu+1)=0$ and hence $\mu=-\frac{1}{2}$. Therefore in this case $\lambda=0$, $\mu=-\frac{1}{2}$, i.e. $\langle X,Y\rangle=\mathcal{K}(X,Y)-\frac{1}{2}(X,JY)$.

3. $\mu \neq 0$, $\lambda \neq 0$. Then $3\mu + 2 = 0$, i.e. $\mu = -\frac{2}{3}$, $\lambda^2 = \frac{1}{9}$; $\lambda = \pm\frac{1}{3}$. So, in this case $\lambda = \pm\frac{1}{3}$, $\mu = -\frac{2}{3}$, i.e. $\langle X, Y\rangle = \mathcal{K}(X, Y) + \frac{1}{3}(X, IY) - \frac{2}{3}(X, JY)$ or $\langle X, Y\rangle = \mathcal{K}(X, Y) - \frac{1}{3}(X, IY) - \frac{2}{3}(X, JY)$.

Inversely, if $\lambda = \mu = 0$, then $A = B = -\frac{1}{4}$, $C = D = 0$ and hence $r = \frac{1}{4}\,\mathrm{id}$. If $\lambda = 0$, $\mu = -\frac{1}{2}$, then $A = B = -\frac{5}{24}$, $C = D = 0$, $r = \frac{5}{18}\,\mathrm{id}$. Finally, if $\lambda = \pm\frac{1}{3}$, $\mu = -\frac{2}{3}$, then $A = B = -\frac{1}{6}$, $C = D = 0$, $r = \frac{3}{8}\,\mathrm{id}$.

So, we proved the following theorem giving a new method of building concrete Einsteinian metrics:

Theorem 22. *Let G be semisimple Lie group equipped with Killing metric $\mathcal{K}$. Then canonical almost antiquaternionic structure on manifold $G \times G$ defines a two-parameter family of pseudo-Riemannian metrics*

$$\langle X, Y\rangle = \mathcal{K}(X, Y) + \lambda\mathcal{K}(X, IY) + \mu\mathcal{K}(X, JY); \quad \lambda, \mu \in \mathbf{R}; \quad \lambda^2 + \mu^2 \neq 1.$$

The metrics are Einsteinian in the following four cases only:

1) $\lambda = \mu = 0$; $\qquad \langle X, Y\rangle = \mathcal{K}(X, Y)$;
2) $\lambda = 0, \mu = -\frac{1}{2}$; $\qquad \langle X, Y\rangle = \mathcal{K}(X, Y) - \frac{1}{2}\mathcal{K}(X, JY)$;
3) $\lambda = \frac{1}{3}, \mu = -\frac{2}{3}$; $\qquad \langle X, Y\rangle = \mathcal{K}(X, Y) + \frac{1}{3}\mathcal{K}(X, IY) - \frac{2}{3}\mathcal{K}(X, JY)$;
4) $\lambda = -\frac{1}{3}, \mu = -\frac{2}{3}$; $\qquad \langle X, Y\rangle = \mathcal{K}(X, Y) - \frac{1}{3}\mathcal{K}(X, IY) - \frac{2}{3}\mathcal{K}(X, JY)$. □

Corollary. *Proper nearly Kaehlerian manifold $(G \times G, \mathcal{J}_0, \langle\cdot,\cdot\rangle_0)$ that is built according to Theorem 21 for semisimple Lie group $M = G$ by Killing metric is Einsteinian manifold.* □

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